CONDITIONAL LOWER BOUNDS ON THE DISTRIBUTION OF CENTRAL VALUES: THE CASE OF MODULAR FORMS

Abstract. Radziwiłł and Soundararajan unveiled a connection between low-lying zeros and central values of L-functions, which they instanciated in the case of quadratic twists of an elliptic curve. This note addresses the case of the family of modular forms in the level aspect.

1. Introduction

Studying L-functions is of utmost importance in number theory at large. Two of their attached data carry critical information: their zeros, which govern the distributional behavior of underlying objects; and their central values, which are related to invariants such as the class number of a field extension or the rank of an elliptic curve. We refer to [3] and references therein for further hindsight.

The spacings of zeros of families of L-functions are well-understood: they are distributed along a universal law, independent of the exact family under consideration, as proven by Rudnick and Sarnak. This recovers the behavior of spacings between eigenangles of the classical groups of random matrices. However, distribution of *low-lying* zeros attached to every reasonable family of L-functions does depend upon the specific setting under consideration. See [7] for a discussion in a general setting.

More precisely, let $L(s, f)$ be an L-function attached to an arithmetic object f. Consider its nontrivial zeros written in the form $\rho_f = \frac{1}{2} + i\gamma_f$ where γ_f is a priori a complex number. We renormalize the mean spacing of the zeros to 1 by setting $\tilde{\gamma}_f = \log c(f)\gamma_f/2\pi$. Let h be an even Schwartz function on $\mathbb{R}$ whose Fourier transform is compactly supported, in particular it admits an analytic continuation to all $\mathbb{C}$. The one-level density attached to f is defined by

$$D(f, h) = \sum_{\gamma_f} h\left(\tilde{\gamma}_f\right). \tag{1.1}$$

The analogy with the behavior or small eigenangles of random matrices led Katz and Sarnak to formulate the so-called *density conjecture*, claiming the same universality for the types of symmetry of families (understood in a reasonable sense, see [7]) of L-functions as those arising for classical groups of random matrices.

Conjecture 1 (Katz-Sarnak). *Let $\mathcal{F}$ be a family of L-functions, and $\mathcal{F}_X$ a finite truncation increasing to $\mathcal{F}$ when X grows. Then for all even Schwartz function on $\mathbb{R}$ with compactly supported Fourier transform, there is one classical group G among* U, SO(even), SO(odd), O *or* Sp *such that*

$$\frac{1}{|\mathcal{F}_X|} \sum_{f \in \mathcal{F}_X} D(f, h) \xrightarrow[X\to\infty]{} \int_{\mathbb{R}} h(x) W_G(x) dx, \tag{1.2}$$

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where $W_G(x)$ is the explicit distribution function modeling the distribution of the eigenangles of the corresponding group of random matrices[1]. *The family $\mathcal{F}$ is then said to have the* type of symmetry *of G.*

This distribution of central values are also finely understood, and the Keating-Snaith conjecture predicts that the logarithmic central values $\log L(\frac{1}{2}, f)$ are asymptotically distributed according to a normal distribution, with explicit mean and variance depending on the family.

Conjecture 2 (Keating-Snaith)**.** *For any positive real numbers $\alpha < \beta$, there is a mean $M_{\mathcal{F}}$ and a variance $V_{\mathcal{F}}$ such that*

$$\frac{1}{|H_k(q)|}\left|\left\{f \in H_k(q) \ : \ \frac{\log L(\frac{1}{2}, f) - M_{\mathcal{F}}}{V_{\mathcal{F}}} \in (\alpha, \beta)\right\}\right| \xrightarrow[X\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx, \tag{1.3}$$

when X grows to infinity. In particular, the family of the logarithmic central values $L(\frac{1}{2}, f)$ equidistributes asymptotically with respect to a normal distribution.

Radziwiłł and Soundararajan [6] established a general principle that any restricted result towards Conjecture 1, which in particular implies lower bound on the non-vanishing of central L-values, can be refined to show that most such L-values have the typical size predicted by Conjecture 2. They instanciated this technique in the case of quadratic twists of a given elliptic curve and suggested the wide applicability of this approach, in particular in the case of modular forms building on the pioneering work of Iwaniec, Luo and Sarnak [2]. This short note explains how to do so in the case of modular forms in the level aspect.

More precisely, for integers $k \geqslant 2$ and $q \geqslant 1$, let $H_k(q)$ be an orthogonal basis of primitive Hecke eigenforms, which is a basis of the space of newforms $S_k^{\text{new}}(q)$. We let $c(f) = k^2 q$ the analytic conductor of f. Introduce for a general sequence $(a_f)_{f\in H_k(q)}$ the harmonic average

$$\sum_{f\in H_k(q)}^h a_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in H_k(q)} \frac{a_f}{\|f\|^2} \tag{1.4}$$

which includes the suitable weights in order to apply the Petersson trace formula. In this setting, the seminal work of Iwaniec, Luo and Sarnak [2] and the recent achievement of Baluyot, Chandee and Li [1] obtain the following restricted statement towards Conjecture 1.

Theorem 1 (Iwaniec, Luo, Sarnak & Baluyot, Chandee, Li)**.** *For any smooth function Ψ compactly supported and any Schwartz function Φ such that its Fourier transform $\hat{\Phi}$ is supported into $(-4, 4)$, we have*

$$\lim_{Q\to\infty} \frac{1}{N(Q)} \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h D(f, \Phi) \longrightarrow \int_{\mathbb{R}} W_O \Phi = \hat{\Phi}(0) + \frac{1}{2}\Phi(0), \tag{1.5}$$

where $W_O = 1 + \frac{1}{2}\delta_O$ is the orthogonal density and $N(Q)$ is the weighted cardinality of the family, i.e.

$$N(Q) = \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h 1. \tag{1.6}$$

[1] See e.g. [2] for the precise formulas of these density functions.

Building on this result and exploiting the methodology outlined by Radziwiłł and Soundararajan, we prove the following statement towards Conjecture 2.

Theorem 2. *Let $H_k(q)$ be a basis of modular Hecke eigenforms of weight $k \geqslant 2$, level $q \geqslant 1$ and trivial nebentypus. We have, for any $\alpha < \beta$,*

$$\frac{1}{N(Q)}\left|\left\{f \in H_k(q) \ : \ \frac{\log L(\frac{1}{2}, f) - \frac{1}{2}\log\log c(f)}{\sqrt{\log\log c(f)}} \in (\alpha, \beta)\right\}\right| \geqslant \frac{5}{8}\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx + o(1). \quad (1.7)$$

Note that [6] mentions the analogous result in the weight aspect on average over the weight, for the full modular group.

A forthcoming work explores this technique for broader classes of functions, with emphasis on the needed structural knowledge in order for it to work.

1.1. **Strategy of proof and structure of the paper.** In Section 2 we recall the needed definitions on modular L-functions. In particular, explicit formulas relate central values of L-functions to sums of modular coefficients over primes, so that most of the study reduces to understanding such sums. In Section 2.4 we establish results on sums of powers of modular coefficients that will be of critical importance in Section 3 where moments of these sums are shown to match the moments of the normal distribution. Section 4 concludes the proof by showing that the extra terms arising in the explicit formula, in the guise of sum over zeros, are negligible except for a small proportion of modular forms.

Remark. Radziwiłł and Soundararajan [6] outline a general strategy to prove such results, but in the specific case they address they rely on the Poisson summation formula to estimate character sums, as well as their complete multiplicativity. These tools are however not as neat in the case of modular coefficients, and it requires the inductive use of Hecke relations and of trace formulas, as recently shown in the work [4] in its study of centered moments of the one-level density.

2. Odds and ends

2.1. **Modular L-functions.** Let f be a holomorphic cusp newform of weight k, level q and trivial nebentypus. It has an attached L-function defined by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \quad (2.1)$$

where $a_f(n)$ is its Fourier coefficient, defined by the Fourier expansion

$$f(z) = \sum_{n \geqslant 1} a_f(n)(4\pi n)^{k/2} e(nz). \quad (2.2)$$

In this normalization, Deligne's bound states that $a(n) \ll d(n) \ll n^\varepsilon$, where $d(n)$ denotes the divisor function. In particular, the Dirichlet series (8) converges for all $\Re(s) > 1$. The L-function $L(s, f)$ can be completed by explicit gamma factors so that we have the functional equation

$$\Lambda(s, f) := \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s + \tfrac{1}{2}) = \varepsilon_f \Lambda(1 - s, f) \quad (2.3)$$

where $\varepsilon_f \in \{\pm 1\}$ is the root number of f. For f a primitive Hecke eigenform, we also have the Euler product

$$L(s,f) = \prod_p (1 - a_f(p)p^{-s} + p^{-2s})^{-1} = \prod_p (1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1} \tag{2.4}$$

where the sum is over prime numbers p, and $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ are called the spectral parameters of f at p. This expression encapsulates the Hecke relations satisfied by the coefficients. By taking the logarithmic derivative of this expression, we obtain

$$-\frac{L'}{L}(s,f) = \sum_{n \geqslant 1} \frac{\Lambda_f(n)}{n^s} \tag{2.5}$$

where $\Lambda_f(n) = (\alpha_f(p)^k + \beta_f(p)^k)\log(p)$ if $n = p^k$ is a prime power, and $\Lambda_f(n) = 0$ otherwise.

2.2. Explicit formula for sums over zeros.

We have the celebrated Weil explicit formula, relating sum over zeros of L-functions to a sum over primes of its spectral parameters (see [2, (4.11)] or [4, (3.2)]):

$$D(f,\Phi) = \hat{\Phi}(0) - \frac{2}{\log c(f)} \sum_p \sum_{\nu \geqslant 1} (\alpha_f(p)^\nu + \beta_f(p)^\nu) \frac{\log p}{p^{\nu/2}} \hat{\Phi}\left(\frac{\nu \log p}{\log c(f)}\right) + O\left(\frac{1}{\log c(f)}\right). \tag{2.6}$$

Using the relations between coefficients and spectral parameters, and the bounds on $a_f(n)$, we obtain that the terms $\nu \geqslant 3$ contribute as an error term, so that we deduce as in [2, Lemma 4.1] the following expansion of the one-level density.

Proposition 1 (Explicit formula for sums over zeros)**.** *We have*

$$D(f,\Phi) = \hat{\Phi}(0) + \tfrac{1}{2}\Phi(0) + P^{(1)}(f,\Phi) + P^{(2)}(f,\Phi) + O\left(\frac{\log\log c(f)}{\log c(f)}\right) \tag{2.7}$$

where, for $\nu \geqslant 1$, we let

$$P^{(\nu)}(f,\Phi) = \frac{2}{\log c(f)} \sum_p a_f(p^\nu) \frac{\log p}{p^{\nu/2}} \hat{\Phi}\left(\frac{\nu \log p}{\log c(f)}\right). \tag{2.8}$$

2.3. Explicit formula for central values.

The connection between central values of L-functions, sums over primes and sums over zeros dates back to Selberg, and can be found in [6, Proposition 1] in the case of quadratic characters. The proof carries on generally.

Proposition 2 (Explicit formula for central values)**.** *Assume that $L(\frac{1}{2}, f)$ is nonzero. We have, for all $x \leqslant c(f)$,*

$$L(\tfrac{1}{2}, f) = P(f,x) - \tfrac{1}{2}\log\log x + O\left(\frac{\log c(f)}{\log x} + \sum_{\gamma_f} \log(1 + (\gamma_f \log x)^{-2})\right) \tag{2.9}$$

where

$$P(f,x) = \sum_{\substack{p<x \\ p\nmid q}} \frac{a_f(p)}{p^{1/2}}. \tag{2.10}$$

2.4. **Sums over primes of coefficients.** A central part of the argument consists in studying the distribution of the sums over primes $P(f, x)$ through the moment methods (see Section 3). In doing so, sums of products of Fourier coefficients will arise, and we give the necessary results here. Inspired by [4], introduce the notation, for any integer $a \geqslant 1$,

$$F(p, a) := \frac{a_f(p)^a}{p^{a/2}}. \tag{2.11}$$

We state in this section the needed estimates for the sums over primes of such powers of coefficients. Informally, the powers $a = 1$ will be small by Perron formula and lower bounds on L-functions, powers $a = 2$ will have a nontrivial size and will contribute ultimately by Rankin-Selberg type results, and higher powers $a \geqslant 3$ will contribute negligibly.

Lemma 1 (Large parts)**.** *We have, for all $a \geqslant 3$,*

$$\sum_{p<x} F(p, a) \ll 1. \tag{2.12}$$

Proof. Using Deligne's bound $a_f(p) \ll 1$, the result is clear since the sum converges absolutely, as does the sum of $p^{-3/2}$. □

Lemma 2 (2-parts)**.** *We have*

$$\sum_{p<x} F(p, 2) = \log\log(x) + O(1). \tag{2.13}$$

Proof. This is a consequence of the Hecke relations and of Rankin-Selberg theory, see for instance [5, Lemma 3]. □

Lemma 3 (1-parts)**.** *We have, for all $n \geqslant 1$,*

$$\sum_{p_i \neq p_j} \prod_{i=1}^{n} F(p_i, 1) \ll 1 \tag{2.14}$$

Proof. This is analogue to [4, Lemma 2.12]. The proof consists in an induction on the number of terms, and boils down to adding the missing primes (showing it is of negligible impact by the above lemmas), and then using a Perron formula to relate the sought sum to L'/L, on which we have bounds that are enough for the result.

Their result [4, Lemma 2.12] reads

$$\sum_{p \leqslant x} \left(b_f(p) := \frac{a_f(p)\log(p)}{p^{1/2}} \right) \ll \log(x) \tag{2.15}$$

and by partial summation, we therefore deduce

$$\sum_{p \leqslant x} \frac{a_f(p)}{p^{1/2}} = \sum_{p \leqslant x} \frac{b_f(p)}{\log p} = \sum_{p \leqslant x} \left(\sum_{p' \leqslant p} b_f(p') \right) \frac{1}{p \log^2 p} \ll \sum_{p \leqslant x} \frac{1}{p \log p} \ll 1 \tag{2.16}$$

giving the desired result. □

Remark. Note that this "rough" bound on the 1-parts will not be sufficient to bound the whole sum over the family, and the harmonic average (in the guide of trace formulas) will have to be fully exploited in order to get enough cancellations. This bound will however be sufficient to address number of cases.

3. Moments

By the above explicit formula, a critical quantity to understand in order to control the distribution of the central values is the sums over primes $P(f,x)$, and this will be investigated by means of the moment method as in [6]. The following result is analogue to [4, Theorem 3.1].

Proposition 3 (Moment property)**.** *We have, for all $k \geqslant 1$,*

$$\frac{1}{N(Q)} \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^{h} P(f,x)^k \Phi\left(\frac{c(f)}{X}\right) = (M_k + o(1)) \log\log(x)^{k/2} \tag{3.1}$$

where we introduced the k-th Gaussian moment

$$M_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^k e^{-x^2/2} dx = \frac{k!}{2^{k/2}(k/2)!}. \tag{3.2}$$

Proof. We follow the strategy of [6, Proposition 3] using the tools developed in [4, Proposition 4.1], adapting it to the specific sum over primes $P(f,x)$ arising in the explicit formula. Expanding the power $P(f,x)^k$ in

$$\sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^{h} P(f,x)^k \Phi\left(\frac{c(f)}{X}\right) \tag{3.3}$$

we are reduced to study sums of the type

$$\sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \Phi\left(\frac{k^2 q}{X}\right) \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^{h} \frac{a_f(p_1)\cdots a_f(p_k)}{\sqrt{p_1\cdots p_k}}. \tag{3.4}$$

Recalling the definition of $F(p,a)$, this expression splits into a sum of sums of the type

$$\sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \Phi\left(\frac{k^2 q}{X}\right) \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^{h} \sum_{\substack{p_1,\ldots,p_\ell \\ p_i\neq p_j}} \prod_{i=1}^{\ell} F(p_i,a_i) \tag{3.5}$$

so that it is sufficient to study these. We split into different cases according to the number of conspiring primes (i.e. the size of the powers a_i), and use the lemmas established in Section 2.4 to treat each part. The cases are:

- Case 1: each power is at least 2, at least one is larger
- Case 2: each power a_i is 2
- Case 3: exactly one power is 1
- Case 4: at least two powers are 1, but not all
- Case 5: each power is 1

In what follows, the sums are understood as consisting in different primes, otherwise explicitly stated (maybe add a "total" or a sign to emphasize it, when we add primes; or better a prime when the primes are different: if they are equal, merge them in a single $F(p,a+b)$).

Case 1: each power is at least 2*, at least one being larger than* 2. By Lemma 1, we have

$$\sum_{p<x} F(p,3) = O(1). \tag{3.6}$$

Hence, in the the *unrestricted* sum (**??**) over primes (i.e. removing the distinctness condition) each term with a power 2 contributes as $\log\log(x)$ by Lemma 2, and the sums with powers larger than 2 will contribute as $O(1)$. All in all, the whole contribution will be $o(\log\log(x)^{k/2})$.

The cost of removing the distinctness condition is given by factors of the type (**??**) with some primes subject to being equal instead of different. This amounts to increasing the powers, since the very definition implies the property $F(p,a)F(p,b) = F(p,a+b)$, and reducing the number of factors, so that the result follows inductively.

Case 2: each power is 2. By Lemma 2, we have

$$\sum_{p<x} F(p,2) = \sum_{p<x} \frac{a_f(p)^2}{p} = \log\log(x) + O(1). \tag{3.7}$$

Hence, taking the product of all pairs of primes, we get a contribution of $(1+o(1))\log\log(x)^{k/2}$ for the *unrestricted* sum over primes.

The cost of removing the distinctness condition is given by similar sums with extra equality conditions, so that some of the factors will have a power larger than 2, thus falling into Case 1, hence negligible compared to the main contribution of the unrestricted sum.

Case 3: Exactly one 1. We prove this case by induction. If $k = 2$, we cannot be in case 3, so this is empty. If $k = 3$, then case 3 arises with the partition $3 = 1 + 2$, i.e. we have to bound the sum

$$S = \frac{1}{N(Q)} \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^{h} \sum_{p_1} F(p_1,2) \sum_{p_2\neq p_1} F(p_2,1). \tag{3.8}$$

We add the missing primes at a negligible cost of $\sum_p F(p,3) \ll 1$ by Lemma 1. We then expand, bound the 2-part by Lemma 2 which gives $\log\log(x)$, and bound the 1-part by Lemma 3, which is $O(1)$, so that we obtain a contribution bounded by $\log\log(x) = o(\log\log(x)^{k/2})$.

We now assume that $k \geqslant 4$, and assume inductively the properties for smaller values. Since $k \geqslant 4$, there is at least one power larger than 2, or at least two powers 2. We will address both cases separately.

Consider the case when there is a part larger than 2. In the unrestrict sum, the large powers will contribute as $O(1)$ by Lemma 1. Each part with a power 2 will contribute as $\log\log(x)$ by Lemma 2. For the power 1, we use Lemma 3 to conclude, bounding its contribution by a constant. The cost of removing the conspiring primes is smaller, since it will increase the powers and reduce the number of factors, hence falling into the induction hypothesis.

Consider the case when there are at least two parts having a 2. For the unrestricted sum, induction on the number k' of 2-parts shows that the product is of size $\log\log(x)^{k'/2}$, but the 1-part will give a smaller contribution by the Lemma 3. The cost of removing the equal primes is even smaller, since it only increases the powers and reduces the number of terms, exactly as above. This concludes the induction.

Case 4: at least two 1's, but not all. We use Hölder inequality as in [4, Lemma A.3] and induction. Each 2-part contributes as $\log\log(x)$ as above. For the two 1-parts, we add back the terms $p_i = p_j$ in order to have a full sum: this is negligible by induction as above, since it only increases powers and reduces the number of terms.

The remaining full sum is amenable to Hölder inequality and moment bounds. This is the analogue of [4, Theorem 3.1] and this has to be assumed as the induction hypothesis: for all $k \geqslant 1$,

$$\frac{1}{N(Q)} \sum_{q \geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f \in H_k(q)}^{h} P(f,x)^k = (M_k + o(1)) \log\log(x)^{k/2}. \tag{3.9}$$

Proof. We follow the strategy of [4, Lemma 3.7]. We use induction on $k \geqslant 1$. The base case, where only one 1-part will remain: is exactly case 5, which will be dealt with below (independently of these proofs). Assume inductively that the proposition holds for all $k' < k$.

First, we apply Lemma 2 to address the 2-parts and pick half as many factors $\log\log(x)$, and Lemma 1 to address the large parts which do not grow. We are therefore left to bound a sum of the form

$$\frac{1}{N(Q)} \sum_{q \geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f \in H_k(q)}^{h} D(x) \sum_{p_i \neq p_j} \prod_i F(p_i, 1), \tag{3.10}$$

where $D(x)$ is the product of the terms containing powers larger than 1: it has a negligible contribution by the above three cases, so that it suffices to show that the expression with 1's is bounded. The 1-parts have to be formally separated, this will be done *via* Hölder inequality, what will induce extra powers; justifying the need of the "moment type" property used as induction hypothesis.

We add the missing primes $p_i = p_j$, analogously to [4, Lemma A.4]. Letting k be the number of 1-parts and writing

$$\sum_{\substack{p_i \neq p_j \\ 1 \leqslant i \neq j \leqslant k}} = \sum_{\substack{p_i \neq p_j \\ 2 \leqslant i \neq j \leqslant k}} - \sum_{i=2}^{k} \sum_{\substack{p_i \neq p_j \\ p_1 = p_i \\ 2 \leqslant i \neq j \leqslant k}} = \sum_{\substack{p_i \neq p_j \\ 2 \leqslant i \neq j \leqslant k}} -(k-1) \sum_{\substack{p_i \neq p_j \\ p_1 = p_2 \\ 2 \leqslant i \neq j \leqslant k}} . \tag{3.11}$$

In this resulting right hand side, the first sum has one less term and is hence amenable to the induction hypothesis, while the second sum has the terms $F(p_1, 1)$ and $F(p_2, 1)$ colliding into $F(p_1, 2)$ since $p_1 = p_2$, hence also amenable to the induction hypothesis.

Now that we added the missing primes, we need to show the negligibility of

$$S' := \frac{1}{N(Q)} \sum_{q \geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f \in H_k(q)}^{h} \sum_{p_i} F(p_i, 1). \tag{3.12}$$

We use Hölder inequality to separate the sums over the spectral family, with Miller's choices for the exponents ξ_i. We have

$$\begin{aligned} S' &\leqslant \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h \prod_i \left| \sum_{p_i} F(p_i,1) \right| N(Q)^{-\xi_i} \\ &\ll \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h (2,3parts) \prod_i \left(|F(p_i,1)|^{\xi_i^{-1}} N(Q)^{-1} \right)^{\xi_i} \\ &\ll \prod_i \left(\sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h |F(p_i,1)|^{\xi_i^{-1}} N(Q)^{-1} \right)^{\xi_i} \end{aligned}$$

and this last sum is exactly a moment of $F(p_i, a)$ (with the specific choices of ξ_i done by Miller, so that ξ_i^{-1} are indeed integers smaller than k), with less terms that we took out in the 2-parts and large parts. Hence, the induction step is complete. □

Case 5: each power is 1. This is the hard part, and the one needing to make use of the harmonic sum over the family, i.e. trace formulas. We have to prove the following bound on the 1-parts:

$$\frac{1}{N(Q)} \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h \sum_{p_i\neq p_j} \prod_i F(p_i,1) = o(\log\log(x)^{k/2}). \tag{3.13}$$

The strategy is as follows: we complete the sum over newforms $f \in H_k(q)$ as a full spectral sum, including oldforms, as in [4, Lemma 2.6] (see (4.2) therein) or [2], in order to apply trace formulas. The rough bounds given in Lemma 3 allows to truncate the resulting sum as in [4, Section 4.1] up to an error term. We can add the primes dividing the level at the cost of an error term too, by means of Petersson trace formula.

The resulting spectral sum is therefore amenable to the Petersson trace formula, that translates it into an arithmetic sum. Swapping summations and changing variables as in [1] leads to a similar arithmetic sum with different functions than J-Bessel. Such a sum can be translated back into a spectral sum, by means of the Kuznetsov trace formula: the sum is now over different levels, but include the whole spectrum. The resulting function in our case satisfy exactly the same bound as in [4, Lemma 4.3], and they can be bounded *mutatis mutandis.* □

As in [6], this essentially allows to say that the $P(f,x)$, hence the central values, mimicks the behavior of a normal distribution, in phase with the Keating-Snaith conjecture. We encapsulate in the following statement the distributional consequence of this moment method:

Proposition 4. *We have, for all sequence $(a_f)_{f\in H_k(q)}$,*

$$\sum_{\substack{f\in H_k(q) \\ P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)}} a_f = (M(\alpha,\beta)+o(1)) \sum_{f\in H_k(q)} a_f. \tag{3.14}$$

Proof. Asymptotically, Proposition 3 proved that the k-th moment of $P(f,x)$ behaves as the k-th moment of the normal distribution, i.e. when X grows to infinity,

$$\sum_{f\in H_k(q)} P(f,x)^k \Phi\left(\frac{c(f)}{X}\right) \sim \int_{\mathbb{R}} x^k e^{-x^2/2} dx \sum_{f\in H_k(q)} \Phi\left(\frac{c(f)}{X}\right) \tag{3.15}$$

so we deduce that, for any polynomial $Q \in \mathbb{R}[X]$,

$$\sum_{f\in H_k(q)} Q(P(f,x))\Phi\left(\frac{c(f)}{X}\right) \sim \int Q(x) e^{-x^2/2} dx \sum_{f\in H_k(q)} \Phi\left(\frac{c(f)}{X}\right) \tag{3.16}$$

and, by approximating the characteristic function $\mathbf{1}_{(\alpha,\beta)}$ in L^1-norm by a polynomial Q, we deduce that

$$\sum_{\substack{f\in H_k(q)\\ P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)}} a_f = \sum_{f\in H_k(q)} \mathbf{1}_{(\alpha,\beta)}(P(f,x)) a_f \tag{3.17}$$

$$\sim \int_{\mathbb{R}} \mathbf{1}_{(\alpha,\beta)}(x) e^{-x^2/2} dx \sum_{f\in H_k(q)} a_f = M(\alpha,\beta) \sum_{f\in H_k(q)} a_f \tag{3.18}$$

as claimed. □

A similar result has to be available when weighted by one-level densities (analogously to [6, Proposition 3, second part]):

Proposition 5 (Weighted moments property)**.** *We have*

$$\frac{1}{N(Q)} \sum_{q\geqslant 1} \Psi\left(\frac{q}{Q}\right) \sum_{f\in H_k(q)}^h P(f,x)^k D(f,h) \Phi\left(\frac{c(f)}{X}\right) = (M_k + o(1)) \log\log(x)^{k/2} \int_{\mathbb{R}} W_O h. \tag{3.19}$$

In other words, this proposition means that we can decouple the one-level density statement and the moment property, both exploiting trace formulas. The proof uses different mechanics than [6], since they instead use Poisson formula for the sum of characters, since they have complete multipliciativity. We need to make a finer analysis as in [4] and [1].

Proof. Expand the power $P(f,x)^k$ so that we have to deal with sums of the form

$$\sum_{p_i<x} \sum_{q, f\in H_k(q)} \frac{a_f(p_1)\cdots a_f(p_k)}{\sqrt{p_1\cdots p_k}} D(f,h)\Phi\left(\frac{c(f)}{X}\right) \tag{3.20}$$

The innermost one-level density is understood by Proposition 1 and can be written as

$$D(f,h) = \hat{h}(0) + \tfrac{1}{2}h(0) + P^{(1)}(f,x) + P^{(2)}(f,x) + o(\dots) \tag{3.21}$$

Note that $\hat{h}(0) + \frac{1}{2}h(0) = \int hW_O$ is the limiting one-level density. The constant part in this expression can therefore be pulled out of the sum, and the Proposition 3 therefore is applicable as it stands and gives a contribution of

$$(M_k + o(1)) \log\log(x)^{k/2} \int_{\mathbb{R}} W_O h \tag{3.22}$$

so that it remains to prove that the remaining contributions are negligible. We therefore have to understand sums of the form

$$\sum_{p_i<x}\sum_{q,f\in H_k(q)}\frac{a_f(p_1)\cdots a_f(p_k)}{\sqrt{p_1\cdots p_k}}P^{(\nu)}(f,x)\Phi\left(\frac{c(f)}{X}\right) \tag{3.23}$$

where $\nu\in\{1,2\}$. We are hence essentially adding an extra coefficient since $P^{(\nu)}(f,x)$ is a weighted sum of coefficients. The contribution therefore adds a coefficient $a_f(p^\nu)$ for $\nu\in\{1,2\}$. For $\nu=1$, this can be bounded by one of the cases 3, 4 or 5, all negligible in the above proposition. For $\nu=2$, this enters into one of the above cases, which are all negligible except in the case where all the powers are 2, case in which the contribution of $P^{(2)}$ is in fact of constant size, see [4, Lemma 2.9]. For $\nu=1$, this falls in cases 3, 4 or 5. □

4. Proof of the theorem

The above tools being now at hand, we follow the strategy presented in [6]. We will show that there are not many small zeros by an amplification process, which will be used to prove that the sum over zeros in the explicit formula (16) contributes as an error term. The moment method will then allow to select the values for which we are in the desired range, giving the result.

4.1. Amplification of small zeros.

The following result, analogue of [6, Lemma 1], uses the "moment method" to quantify the proportion of $f\in\mathcal{F}$ such that $P(f,x)$ falls into a specific range; and the low-lying zeros result to jointly quantify the proportion of $f\in H_k(q)$ having not too many small zeros.

Proposition 6. *The smooth averaged number of $f\in H_k(q)$ such that $P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)$ and such that there are no zeros with $|\gamma_f|\leqslant(\log X\log\log X)^{-1}$ is larger than*

$$\frac{5}{8}M(\alpha,\beta)N(Q) \tag{4.1}$$

where

$$M(\alpha,\beta)=\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}x^k dx. \tag{4.2}$$

Proof. Choose for h the explicit Féjer kernel up to the maximal Fourier support $L=4$ allowed by the low-lying zero result given in Theorem 5, i.e.

$$h_0(x):=\left(\frac{\sin\pi x}{\pi x}\right)^2 \qquad \hat{h}_0(y)=\max(1-|y|,0), \tag{4.3}$$

which has Fourier transform supported in $(-1,1)$, and $h(x)=h_0(4x)$ so that $\hat{h}(y)=\frac{1}{4}\hat{h}_0(x/4)$ is compactly supported in $(-4,4)$. Let $H=\sum_{\gamma_f}h(\tilde{\gamma}_f)$ to lighten notations for the duration of the proof. We get from the "moment method", i.e. Proposition 3:

$$\sum_{\substack{f\in H_k(q)\\ P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)}} H\Phi\left(\frac{c(f)}{X}\right)\sim M(\alpha,\beta)\sum_{f\in H_k(q)}H\Phi\left(\frac{c(f)}{X}\right), \tag{4.4}$$

and, by Proposition 5, we get

$$\sum_{f\in H_k(q)} H\Phi\left(\frac{c(f)}{X}\right) \sim \int_{\mathbb{R}} W_O h \sum_{f\in H_k(q)} \Phi\left(\frac{c(f)}{X}\right) = \frac{3}{4}\sum_{f\in H_k(q)} \Phi\left(\frac{c(f)}{X}\right), \tag{4.5}$$

where

$$\int W_O(y)h(y)dy = \frac{3}{4} \tag{4.6}$$

by the explicit choice of h, which is allowed to have Fourier support in $(-4,4)$. See [2] for the proof of the optimality of this function in such a setting.

We can now use the similar amplification argument as in Radziwiłł and Soundararajan approach. Rewrite the above sum as

$$\sum_{(\alpha,\beta)} H\Phi = \sum_{\substack{(\alpha,\beta)\\ \exists}} H\Phi + \sum_{\substack{(\alpha,\beta)\\ \nexists}} H\Phi \tag{4.7}$$

where we introduced the following notations, letting $\ell = (\log X \log\log X)^{-1}$,

$$\sum_{(\alpha,\beta)} H\Phi = \sum_{f\in H_k(q)} H\Phi\left(\frac{c(f)}{X}\right)\mathbf{1}_{P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)} \tag{4.8}$$

$$\sum_{\substack{(\alpha,\beta)\\ \exists}} H\Phi = \sum_{\substack{f\in H_k(q)\\ \exists\ |\gamma_f|\leqslant\ell}} H\Phi\left(\frac{c(f)}{X}\right)\mathbf{1}_{P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)} \tag{4.9}$$

$$\sum_{\substack{(\alpha,\beta)\\ \nexists}} H\Phi = \sum_{\substack{f\in H_k(q)\\ \nexists\ |\gamma_f|\leqslant\ell}} H\Phi\left(\frac{c(f)}{X}\right)\mathbf{1}_{P(f,x)/\sqrt{\log\log x}\in(\alpha,\beta)} \tag{4.10}$$

The weights $h(\gamma_f)$ are always non-negative, since the function h we chose is non-negative. If $L(s,f)$ has a zero γ_f smaller than ℓ, then $\tilde{\gamma}_f$ is smaller than $\log\log(X)^{-1}$, and its conjugate is also a zero which has same module. Choosing a continuous function h such that $h(0)=1$, when x grows to infinity $h(\tilde{\gamma}_f)$ is larger than $1-\varepsilon$. We can therefore write

$$\sum_{(\alpha,\beta)} H\Phi = \sum_{\substack{(\alpha,\beta)\\ \exists}} H\Phi + \sum_{\substack{(\alpha,\beta)\\ \nexists}} H\Phi \geqslant (2-\varepsilon)\sum_{\substack{(\alpha,\beta)\\ \exists}} \Phi + \sum_{\substack{(\alpha,\beta)\\ \nexists}} H\Phi = (2-\varepsilon)\sum_{(\alpha,\beta)}\Phi + \sum_{\substack{(\alpha,\beta)\\ \nexists}}(H-2)\Phi. \tag{4.11}$$

On the other hand, the above consequences of the moment method and of the limiting one-level density result given in Proposition 5 rephrase as

$$\sum_{(\alpha,\beta)} \Phi \sim M(\alpha,\beta)\sum_{f\in H_k(q)}\Phi \tag{4.12}$$

$$\sum_{(\alpha,\beta)} H\Phi \sim \frac{3}{4}M(\alpha,\beta)\sum_{f\in H_k(q)}\Phi \tag{4.13}$$

relating the restricted sums to the corresponding whole sums. We therefore deduce

$$(2-\varepsilon)M(\alpha,\beta)\sum_{f\in H_k(q)}\Phi \leqslant \sum_{(\alpha,\beta)} H\Phi - \sum_{\substack{(\alpha,\beta)\\ \nexists}}(H-2+\varepsilon)\Phi \sim M(\alpha,\beta)\tfrac{3}{4}\sum_{f\in H_k(q)}\Phi - \sum_{\substack{(\alpha,\beta)\\ \nexists}}(H-2+\varepsilon)\Phi \tag{4.14}$$

so that, since $0 \leqslant h \leqslant 1$,

$$\tfrac{5}{4} M(\alpha,\beta) \leqslant \sum_{\substack{(\alpha,\beta)\\ \nexists}} (2 - H - \varepsilon)\Phi \leqslant (2-\varepsilon) \sum_{\substack{(\alpha,\beta)\\ \nexists}} \Phi, \tag{4.15}$$

from where we can lower bound the smoothed quantity of $f \in H_k(q)$ having zeros of moduli smaller than ℓ, viz.

$$\sum_{\substack{(\alpha,\beta)\\ \nexists}} \Phi\left(\frac{c(f)}{X}\right) \geqslant (\tfrac{5}{8} - \varepsilon) M(\alpha,\beta) \sum_{f\in H_k(q)} \Phi\left(\frac{c(f)}{X}\right), \tag{4.16}$$

for all $\varepsilon > 0$, as wanted. □

Remark. The constant is exactly the one appearing in the theorem, and this is where we see that the quality of the results towards the density conjecture, i.e. the width of the allowed Fourier support, conditions the quality of this lower bound. Note that this gives the same value as the method in [2] to obtain lower bounds for nonvanishing, as anticipated by [6].

4.2. **Few zeros contributing a lot.** The following proposition is the analogue of [6, Lemma 2], and quantifies how rare are the $f \in H_k(q)$ such that the contribution from the sum over zeros is large.

Proposition 7. *The number of $f \in H_k(q)$ such that*

$$\sum_{|\gamma_f| \geqslant (\log X \log\log X)^{-1}} \log(1 + (\gamma_f \log x)^{-2}) \geqslant \log\log\log(X)^3 \tag{4.17}$$

is $\ll X/\log\log\log X$.

Proof. The same proof as in [6, Lemma 2] holds *mutatis mutandis.* □

4.3. **Conclusion.** This closely follows the argument of [6], now that all the corresponding estimates have been established. We write it here for the sake of completeness. Recall from Proposition 2, with $x = c(f)$, that

$$\log L(1/2, f) = P(f,x) - \tfrac{1}{2}\log\log(x) + O\left(\sum_{\gamma_f} \log(1 + (\gamma_f \log x)^{-2})\right). \tag{4.18}$$

By Proposition 6, we may select f's such that $P(f,x)/\sqrt{\log\log X} \in (\alpha,\beta)$ and that there is no small zeros, without loosing at least a proposition of $\frac{5}{8}$ of the whole family, i.e.

$$\sum_{\substack{f\in H_k(q)\\ P(f,x)/\sqrt{\log\log x} \in (\alpha,\beta)\\ \nexists |\gamma_f| \leqslant (\log X \log\log X)^{-1}}} 1 \geqslant \tfrac{5}{8} M(\alpha,\beta) N(Q). \tag{4.19}$$

By Proposition 7, we may remove f's such that the sum over zeros larger than $(\log X \log\log X)^{-1}$ contributes more than $\log\log\log(X)^3$, since they are asymptotically a negligible cardinality. and the other ones do not contribute that much.

The proportion of f such that $P(x, f)/\sqrt{\log\log c(f)}$ falls into (α, β) is therefore asymptotically larger than $\frac{5}{8}M(\alpha, \beta)$ as claimed in the theorem, henceforth ending the proof.

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