Maass forms and quantum unique ergodicity on decreasing hyperbolic disks

Ivan Doubovik

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Contents

Introduction

One can consider the upper half plane $\mathcal{H}$ and the group $SL_2(\mathbb{Z})$ which is acting on $\mathcal{H}$. On $\mathcal{H}$ we can also introduce a differential structure and look at the Laplacian Δ acting on such a space as a linear operator. The eigenvectors of Δ are called Maass forms and there are 2 main types of Maass forms: Maass cusp forms and Eisenstein series. By defining a measure μ on $\mathcal{H}$ for a Maass cusp form u we can consider the complex measure $u^2\mu$ and for Eisenstein series E we can consider the measure $|E|^2\mu$. The first question we may ask ourselves is how these measures behave asymptotically as their eigenvalue gets large. In the paper by Luo and Sarnak [5] the behavior of $|E|^2\mu$ is discussed for compact Jordan measurable sets. It was then done in the general setting by combining the work of Lindenstrauss and Soundararajan.

Here we are interested in a similar problem, we wish to consider how these measures behave asymptotically when evaluated on a decreasing family of discs. However we shall see that we will need some kind of link between the behavior of the radius of the disks and the eigenvalues in order to make sure that their behavior is not too wild. We will be following the work of Matthew P. Young in his article [8].

1 Maass wave forms

Consider the space $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ which is known as the upper half plane. On such a space we can consider the following metric:

$$\begin{array}{rccl} m: & \mathcal{H} & \longrightarrow & \left\{f \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}} \;\middle|\; f \text{ is a blinear map}\right\} \\ & z & \longmapsto & \begin{array}{rccc} m(z): & \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \frac{\langle x,y\rangle}{\text{Im}(z)^2} \end{array} \end{array}$$

One can check that this makes $\mathcal{H}$ a Riemannian manifold, and that more precisely that $\mathcal{H}$ is a hyperbolic space. We can also show that the geodesics of $\mathcal{H}$ consist of vertical euclidean lines and circles centered on the real line. On $\mathcal{H}$ by considering λ to be the Lebesgue measure we can define the measure $\mu = \frac{1}{y^2}\lambda \otimes \lambda$ more commonly denoted as $\frac{dxdy}{y^2}$ and for this measure can define a bilinear product $\langle \cdot, \cdot \rangle$. For more details about the geometry of $\mathcal{H}$ one can check the following book [1].

For this geometry there is also a natural differential operator that one can consider:

$$\Delta = y^2 \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right)$$

Δ is known as the Laplacian over $\mathcal{H}$. On $\mathcal{H}$ one can also consider the action of the group $SL_2(\mathbb{Z})$ given by:

$$\begin{array}{rccc} \cdot: & SL_2(\mathbb{Z}) \times \mathcal{H} & \longrightarrow & \mathcal{H} \\ & (\gamma, z) & \longmapsto & \frac{az+b}{cz+d} \end{array}$$

This is an action that is natural for the space $\mathcal{H}$ in the sens that it preserves the metric. Now we can consider functions on $\mathcal{H}$ with values in $\mathbb{C}$ that are eigenvectors of Δ, however this set of functions is too large for us to study which is why we introduce the following definition.

Definition 1.1 :

Let $f \in \mathcal{C}_c^\infty(\mathcal{H})$ we shall say that f is a Maass wave form if and only if the following properties are verified:

1. $\forall \gamma \in SL_2(\mathbb{Z}), \forall z \in \mathcal{H}, f(\gamma z) = f(z)$
2. $\exists \lambda \in \mathbb{C}, -\Delta f = \lambda f$
3. $\exists K \geqslant 0, \exists N \geqslant 0, \forall x \in \mathbb{R}, \forall y \in [1; +\infty[, |f(x+iy)| \leqslant K |y|^N$

Since $SL_2(\mathbb{Z})$ contains the unit translation we know that every Maass form has a Fourier expansion, and by putting this expansion in the equation $-\Delta f = \lambda f$ we obtain the following proposition:

Proposition 1.2:

Let f be a Maass wave form such that $-\Delta f = \left(\frac{1}{4}+t^2\right) f$ then it as the following Fourier expansion:

$$f(x+iy) = cy^{\frac{1}{2}-it} + dy^{\frac{1}{2}+it} + y^{\frac{1}{2}}\sum_{n=1}^{+\infty} a_n \, \mathrm{K}_{it}(2\pi ny)\cos(2\pi nx)$$

Where K is the Bessel function and c, d and a_n are complex numbers.

There is a particular type of Maass wave forms called Eisenstein series defined by:

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s$$

Where Γ_∞ is the subgroup generated by the unit translation. And one may check that we have:

$$-\Delta E(\cdot, s) = s(1-s)E(\cdot, s)$$

For a Maass form f we shall say that f is cusp form if it's degree 0 Fourier coefficient is 0 which allows us to prove the following theorem:

Theorem 1.3:

Let f be a Maass cusp form such that:

$$f(x+iy) = y^{\frac{1}{2}}\sum_{n=1}^{+\infty} a_n \, \mathrm{K}_{it}(2\pi ny)\cos(2\pi nx)$$

Then we can define:

$$L(f,s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

Which converges for $\mathrm{Re}(s) > \frac{3}{2}$ and for ε being the parity of f one can define:

$$\Lambda(f,s) = \pi^{-s}\Gamma\left(\frac{s-\varepsilon+it}{2}\right)\left(\frac{s-\varepsilon-it}{2}\right)L(f,s)$$

Then $\Lambda(f,\cdot)$ can be extended to an entire function and satisfies the following equation:

$$\Lambda(f,s) = (-1)^\varepsilon \Lambda(f, 1-s)$$

One can also see that Δ is a self adjoint operator on the space $\mathcal{C}_c^\infty(SL_2(\mathbb{Z})\backslash\mathcal{H})$ and by density it can be extended to a self ajoint operator over the space $L^2(SL_2(\mathbb{Z})\backslash\mathcal{H})$ which is a Hilbert space. Thus we would like to diagonalize this operator but unfortunately this can not be done in a naive way. Indeed this operator is not a compact operator but if instead of just considering a discrete Hilbert base we consider a continuous Hilbert base we can still manage to diagonalize it. More precisely we have the following result for more details the reader can take a look at the following book [3]:

Theorem 1.4:

There exists a orthonormal family $\mathcal{B}$ of Maass cusp forms such that by denoting n the counting measure over over $\mathcal{B}$ we have that:

$$\forall f \in L^2(SL_2(\mathbb{Z})\backslash\mathcal{H}),$$
$$f = \left\langle f, \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \right\rangle + \int\limits_{\mathcal{B}} \langle f, v\rangle \, v dn(v) + \frac{1}{4\pi}\int\limits_{\mathbb{R}} \left\langle f, E(\cdot, \frac{1}{2} + it) \right\rangle E(\cdot, \frac{1}{2} + it) d\lambda(t)$$

2 L-functions

In the previous section, to each Maass cusp f form we were able to associate it's L function defined by $L(f,s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ where a_n are the Fourier coefficients of f. And this concept is not unique to Maass forms, it is also done for modular forms and other objects. So in this section we shall define L-functions in a very general way so that we can obtain results that are true for all of these settings.

2.1 Definitions

Our first task is to define what we consider an L-function to be. The idea behind this is we want it have a Dirichlet series, a Euler product and we would like to be able to complete it into a function which can be extended over $\mathbb{C}$. We also wish this extension to have a so called functional equation. And as in our example with Maass forms the L-function is actually a function of two variables, the function and an element of $\mathbb{C}$.

We will need functions that do not behave too badly, this is why we introduce the following definition.

Definition 2.1:

Let f be an entire function, we shall say that:

$$f \text{ is of order } \leqslant 1 \iff \forall \beta > 1, f(|z|) \underset{|z|\to+\infty}{\ll} \exp(|z|^{\beta})$$

Definition 2.2:

Let f be a meromorphic function over $\mathbb{C}$, we say that:

$$f \text{ has a Dirichlet series}$$
$$\iff$$
$$\exists a \in \mathbb{C}^{\mathbb{N}\setminus\{0\}}, \begin{cases} \forall z \in \mathbb{C}, \mathrm{Re}(z) > 1, f(z) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^z} \\ \forall z \in \mathbb{C}, \mathrm{Re}(z) > 1, \sum_{n\geqslant 1} \left|\frac{a(n)}{n^z}\right| \text{ converges} \end{cases}$$

Definition 2.3:

Let f be a holomorphic function on the set $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1\}$, $p \in \mathcal{P}$ and $d \in \mathbb{N} \setminus \{0\}$ we say that:

$$f \text{ is a } p\text{-term of degree } d \iff \exists a \in B(0,p)^{[\![1,d]\!]}, f(z) = \prod_{n=1}^{d} \frac{1}{1 - a(n)p^{-z}}$$

And let us define $\mathcal{T}_{d,p} = \left\{ f \in \mathbb{C}^{\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1\}} \mid f \text{ is a } p\text{-term of degree } d \right\}$.

Definition 2.4:

Let f be a meromorphic function, we say that:

$$f \text{ has a Euler product of degree } d$$
$$\iff$$
$$\exists T \in \prod_{p \in \mathcal{P}} \mathcal{T}_{d,p}, \begin{cases} f|_{\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1\}} = \prod_{p \in \mathcal{P}} T(p) \\ \forall z \in \mathbb{C}, \operatorname{Re}(z) > 1 \Rightarrow \prod_{p \in \mathcal{P}} T(p)(z) \text{ converges absolutly} \end{cases}$$

By using the properties of absolute converging products we have the following result.

Proposition 2.5:

Let f be a meromorphic function and let $d \in \mathbb{N} \setminus \{0\}$ such that f has a Euler product pf degree d then:

$$\forall z \in \mathbb{C}, \operatorname{Re}(z) > 1 \Rightarrow f(z) \neq 0$$

In the function equation of the L-function for Maass cusp forms we have some gamma factors that appear this is why we introduce the following definitions.

Definition 2.6 :

Let f be a meromorphic over $\mathbb{C}$ defined on an open set U and $d \in \mathbb{N}\setminus\{0\}$, we say that:

$$f \text{ is a gamma product of degree } d$$
$$\Longleftrightarrow$$
$$\exists a \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -1\}^{[\![1,d]\!]}, \begin{cases} \forall z \in \mathbb{C}, \operatorname{Card}(a^{-1}(\{z\})) = \operatorname{Card}(a^{-1}(\{\overline{z}\})) \\ \forall z \in U, f(z) = \pi^{-d\frac{z}{2}} \prod_{n=1}^{d} \Gamma\left(\frac{z+a(n)}{2}\right) \end{cases}$$

Let us also notice that $\forall z \in \mathbb{C}, \operatorname{Card}(a^{-1}(\{z\})) = \operatorname{Card}(a^{-1}(\{\overline{z}\}))$ just means that the numbers $a(n)$ come in conjugate pairs.

In the next definition we introduce a set E, it should be viewed as the set of Maass cusp forms for example.

Definition 2.7 :

Let E be a set, let $\gamma \in \mathbb{C}^{E\times(\mathbb{C}\setminus\{0,1\})}$ and let $d \in \mathbb{N}\setminus\{0\}$ we then define:

γ is a gamma factor of degree $d \iff \forall a \in E, \gamma(a,\cdot)$ is a gamma product of degree d

Definition 2.8 :

Let E be a set, let $\Lambda \in \mathbb{C}^{E\times(\mathbb{C}\setminus\{0,1\})}$, let $a, b \in E$, we say that:

$$\Lambda \text{ satisfies an } (a,b)\text{-functional equation}$$
$$\Longleftrightarrow$$
$$\exists \eta \in \mathcal{S}^1, \forall z \in \mathbb{C}\setminus\{0,1\}, \Lambda(a,z) = \eta\Lambda(b, 1-z)$$

In the following definition the function for $a \in E$, $\phi(a)$ should be viewed as the dual object of a. In the case of Maass forms ϕ is the identity.

***Definition 2.9** (L-function) :*

Let E be a set, let $L \in \mathbb{C}^{E\times(\mathbb{C}\setminus\{0,1\})}$ and let $\phi \in E^E$. Then we say that L is an L-

function over the set E for the dual function $\phi \iff \exists d \in \mathbb{N}\setminus\{0\}$, $\exists \gamma$ a gamma factor of degree d, $\exists c \in \mathbb{N}^E$ such that the following properties are true:

1. $\forall a \in E$, $\exists u \in \mathbb{C}^{\mathbb{N}\setminus\{0\}}$, u is a Dirichlet series for $L(a,\cdot)$ and $\overline{u}$ is a Dirichlet series for $L(\phi(a),\cdot)$.

2. $\forall a \in E$, $L(a,\cdot)$ has a Euler product of degree d.

3. $\forall a \in E$, $\gamma(a,\cdot) = \gamma(\phi(a),\cdot)$ and $c(a) = c(\phi(a))$.

4. $\forall a \in E$, $c(a)^{\frac{\cdot}{2}}\gamma(a,\cdot)L(a,\cdot)$ is holomorphic over $\mathbb{C}\setminus\{0,1\}$ and meromorphic over $\mathbb{C}$ or order $\leqslant 1$ and satisfies an $(a,\phi(a))$-functional equation.

In the previous definition we can check that d, γ and c are unique thus we shall say that (d,γ,c) are the parameters of L. The uniqueness of these parameters justify that the following definition make sense.

Definition 2.10:

Let f be a gamma product of degree d defined on an open set U. And $a \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -1\}^{[\![1;d]\!]}$, $f(z) = \pi^{-d\frac{z}{2}} \prod_{n=1}^{d} \Gamma\left(\frac{z+a(n)}{2}\right)$. We then define the conductor $\operatorname{cond}(f) \in \mathbb{R}^{\mathbb{C}}$ by:

$$\operatorname{cond}(f)(z) = \prod_{n=1}^{d} (|z + a(n)| + 1)$$

Definition 2.11:

Let E be a set, let $\gamma \in \mathbb{C}^{E\times\mathbb{C}\setminus\{0,1\}}$ be a gamma factor of degree d we can then define:

$$\begin{array}{rccl} \operatorname{cond}(\gamma): & E\times\mathbb{C} & \longrightarrow & \mathbb{R} \\ & (a,z) & \longmapsto & \operatorname{cond}(\gamma(a,\cdot))(z) \end{array}$$

Definition 2.12:

Let E be a set, let $L \in \mathbb{C}^{E\times\mathbb{C}\setminus\{0,1\}}$ be an L-function over the set E and let (d,γ,c) be the parameters of L. Then we define the conductor $\operatorname{cond}(L)$ by:

$$\operatorname{cond}(L)(a,z) = c(a)\operatorname{cond}(\gamma)(a,z)$$

Aside form Maass forms a nice example of L-function would be the Riemann zeta function. Indeed one has that:

$$\forall z \in \mathbb{C}, \text{Re}(z) > 1 \Rightarrow \zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-z}}$$

To stick with our notations, we can define a set $E = \{0\}$ and $L(0, z) = \zeta(z)$. And one can show that ζ can be completed into $\xi(z) = \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z)$ which as the functional equation $\xi(z) = \xi(1 - z)$. Thus the conductor q associated to ζ would be $q(z) = |z| + 1$.

Using this abstract definition of L-functions we are able to define operations that create new kinds of L-functions. For example for a set E and $a \in E$ we can define the symmetric square L-functions $L(\text{sym}^2 a, z)$.

2.2 Convexity bounds

Using the associated Dirichlet series we can describe the behavior of an L-function in $\{z \in \mathbb{C} \mid \text{Re}(z) > 1\}$ and using the functional equation given by point 3 of the previous definition we can also describe its behavior in $\{z \in \mathbb{C} \mid \text{Re}(z) < 0\}$. However it is useful to know its behavior on the so called critical strip $\{z \in \mathbb{C} \mid \text{Re}(z) \in [0; 1]\}$ and finding the best possible bounds is an open problem. So we will just give the so called trivial bounds given by some basic complex analysis.

The following lemma is a direct consequence of the fact that L-functions have Dirichlet series and that those converge absolutely.

Lemma 2.13:

Let E be a set, $X \in \mathcal{P}(E \times \mathbb{C})$ and let L be an L-function associated to E. Then:

$$\forall a \in E, \forall x > 1, \exists K \geqslant 0, \forall y \in \mathbb{R}, |L(a, x + iy)| \leqslant K$$

By combining the previous lemma with the fact that L-functions have a functional equation and using Stirling's formula (see Appendix) we obtain the following lemma.

Lemma 2.14:

Let E be a set, let L be an L-function associated to E and let $q = \text{cond}(L)$, then one has:

$$\forall a \in E, \forall x < 0, \exists K \geqslant 0, \forall y \in \mathbb{R}, |L(a, x + iy)| \leqslant K q(a, x + iy)^{\frac{1}{2} - x}$$

Now to find out what happens in the critical strip $\{z \in \mathbb{C} \mid \text{Re}(z) \in [0; 1]\}$ we will use the Phragmén–Lindelöf principle. Thus we shall first take a look at some results from complex analysis.

Lemma 2.15 *(The Phragmen Lindelöf principle)* ***:***

Let $c \geqslant 0$, let $E = \left\{z \in \mathbb{C} \;\middle|\; \mathrm{Re}(z) \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \text{ and } \mathrm{Im}(z) \geqslant c\right\}$ and let $f \in \mathcal{C}(E)$ be a holomorphic function over $\overset{\circ}{E}$ such that $\forall z \in \partial E, |f(z)| \leqslant 1$ and $\exists A, B > 0, \exists \eta \in [0, 1[, \forall z \in E, |f(x+iy)| \leqslant A \exp(B \exp(\eta y))$ then:

$$\forall z \in E, |f(z)| \leqslant 1$$

Proof :

Let $\varepsilon > 0$ and let $\delta = 1 - \eta > 0$.
We can define the following function:

$$\begin{array}{rccl} g: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \exp\left(-\varepsilon \exp\left(-i\left(1-\frac{\delta}{2}\right)z\right)\right) \end{array}$$

One can check that g is a holomorphic function and now we shall study the behavior of g. To do so let $z = x + iy \in \mathbb{C}$ and let us compute the following expression:

$$\exp(iz) = \exp(-y + ix) = \exp(-y)(\cos(x) + i\sin(x))$$

Thus we have that for $a \in \mathbb{R}$:

$$\begin{aligned} \exp(a\exp(iz)) &= \exp(a\exp(-y)(\cos(x) + i\sin(x))) \\ &= \exp(a\exp(-y)(\cos(x)) \exp(ia\exp(-y)\sin(x))) \end{aligned}$$

And this gives us $|\exp(a\exp(iz))| = \exp(a\exp(-y)\cos(x))$ and by applying this to g we obtain that:

$$|g(z)| = \exp\left(-\varepsilon \exp\left(\left(1-\frac{\delta}{2}\right)y\right)\cos\left(-\left(1-\frac{\delta}{2}\right)x\right)\right)$$

And for $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ we have that $\cos\left(-\left(1-\frac{\delta}{2}\right)x\right) \geqslant \cos\left(\left(1-\frac{\delta}{2}\right)\frac{\pi}{2}\right) > 0$, thus be defining $K_1 = \cos\left(\left(1-\frac{\delta}{2}\right)\frac{\pi}{2}\right)$ we have:

$$|g(z)| \leqslant \exp\left(-\varepsilon K_1 \exp\left(\left(1-\frac{\delta}{2}\right)y\right)\right)$$

And for $z \in E$ by noting $z = x + iy$ we can compute:

$$\begin{aligned}
|f(z)g(z)| &\leqslant A\exp\left(B\exp\left((1-\delta)\,y\right)\right)\exp\left(-\varepsilon K_1 \exp\left(\left(1-\frac{\delta}{2}\right)y\right)\right)\\
&= A\exp\left(B\exp\left((1-\delta)\,y\right) - \varepsilon K_1 \exp\left(\left(1-\frac{\delta}{2}\right)y\right)\right)\\
&= A\exp\left(B\exp\left(\left(1-\frac{\delta}{2}\right)y\right)\exp\left(-\frac{\delta}{2}y\right) - \varepsilon K_1 \exp\left(\left(1-\frac{\delta}{2}\right)y\right)\right)\\
&= A\exp\left(B\exp\left(\left(1-\frac{\delta}{2}\right)y\right)\left(\exp\left(-\frac{\delta}{2}y\right) - \varepsilon K_1\right)\right)
\end{aligned}$$

This show us that $\lim\limits_{y\to+\infty} |f(x+iy)g(x+iy)| = 0$. This means that:

$$\exists M \geqslant 0, \forall z \in E, \operatorname{Im}(z) \geqslant M \Rightarrow |f(z)g(z)| \leqslant 1$$

And we can define $F = \{z \in E \mid \operatorname{Im}(z) \leqslant M\}$, this set being compact we can apply the maximum modulus principal. Since we have chosen M such that $\forall z \in \partial F$, $|f(z)g(z)| \leqslant 1$ we have that $\forall z \in F$, $|f(z)g(z)| \leqslant 1$. And we also have that $z \in E \backslash F$, $|f(z)g(z)| \leqslant 1$. So this gives us:

$$\forall z \in E, \forall \varepsilon > 0, \left| f(z)\exp\left(-\varepsilon\exp\left(-i\left(1-\frac{\delta}{2}\right)z\right)\right)\right|$$

Thus for $z \in E$ we have that:

$$\lim_{\varepsilon\to 0}\left| f(z)\exp\left(-\varepsilon\exp\left(-i\left(1-\frac{\delta}{2}\right)z\right)\right)\right| \leqslant 1$$

And since $\lim\limits_{\varepsilon\to 0}\exp\left(-\varepsilon\exp\left(-i\left(1-\frac{\delta}{2}\right)z\right)\right) = 1$ we finally obtain that:

$$|f(z)| \leqslant 1$$

And this gives us the wanted result.

□

__Lemma 2.16 :__

Let $a < b$, let $a \geqslant 0$, let $E = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in [a; b] \text{ and } \operatorname{Im}(z) \geqslant a\}$ and let $f \in \mathcal{C}(E)$ be a holomorphic function over $\overset{\circ}{E}$ such that $\forall z \in \partial E, |f(z)| \leqslant 1$ and $\exists A, B > 0, \exists \eta \in [0, 1[, \forall z \in E, |f(x+iy)| \leqslant A\exp\left(B\exp\left(\eta\frac{\pi}{b-a}y\right)\right)$ then:

$$\forall z \in E, |f(z)| \leqslant 1$$

Using these results we are able to deduce the so called convexity bound for L-functions which is expressed in the following proposition.

Theorem 2.17 *(Convexity bounds)* **:**

Let E be a set, let $L \in \mathbb{C}^{E\times\mathbb{C}\setminus\{0,1\}}$ be an L-function associated to this set and let $q = \text{cond}(L)$. Then we have the following convexity bound:

$$\forall\varepsilon > 0, \forall a \in E, \forall x \in [0,1], \exists M \geqslant 0, \exists K \geqslant 0, \forall y \in \mathbb{R},$$
$$|y| \geqslant M \Rightarrow |L(a, x+iy)| \leqslant Kq(a, x+iy)^{\frac{1-x}{2}+\varepsilon}$$

Proof :

Let $\epsilon > 0$ and $\eta > 0$. We wish to use the previous lemma, to do so let $\alpha = \frac{\frac{1}{2}+\eta}{1+2\eta} \geqslant 0$, let $\beta = \frac{(1+\eta)\left(\frac{1}{2}+\eta\right)}{1+2\eta} \geqslant 0$ and by considering Log as being a logarithm defined on $\mathbb{C}\setminus]-\infty;0]$ we define:

$$\begin{array}{rccl} g: & \mathbb{C}\setminus]-\infty;0] & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \exp(-(-\alpha z+\beta)\,\text{Log}(-iz)) \end{array}$$

One can check that g is holormophic and now we shall take a look at the behaviour of g. To do so consider $z = x+iy \in \mathbb{C}$ and let us calculate:

$$\begin{aligned} g(x+iy) &= \exp\left(-(-\alpha(x+iy)+\beta)\,\text{Log}(-iz)\right) \\ &= \exp\left(-(\beta-\alpha x-i\alpha y)(\ln(|z|)+i\arg(-iz))\right) \\ &= \exp(-((\beta-\alpha x)\ln(|z|)+\alpha y\arg(-iz)+i((\beta-\alpha x)\arg(-iz)-\alpha y\ln(|z|)))) \end{aligned}$$

Thus we obtain that:

$$\begin{aligned} |g(x+iy)| &= \exp(-((\beta-\alpha x)\ln(|z|)+\alpha y\arg(-iz))) \\ &= \exp(-(\beta-\alpha x)\ln(|z|))\exp(-\alpha y\arg(-iz)) \\ &= |z|^{-(\beta-\alpha x)}\exp(-\alpha y\arg(-iz)) \end{aligned}$$

Then one can notice that $\arg(-iz) = \arg(z)+\arg(-i) = \arg(z)-\frac{\pi}{2}$ and for $x, y \geqslant 0$, we have $\arg(x+iy) = \frac{\pi}{2}-\arg(y+ix) = \frac{\pi}{2}-\tan^{-1}\left(\frac{x}{y}\right)$. Which leads us to:

$$|g(x+iy)| = |z|^{-(\beta-\alpha x)}\exp\left(\alpha y\tan^{-1}\left(\frac{x}{y}\right)\right)$$

Now consider $F = \{z \in \mathbb{C} \mid \text{Re}(z) \in [-\eta; 1+\eta] \text{ and } \text{Im}(z) > 0\}$, then one can also check by using a Taylor expansion of $\tan^{-1}$ in 0 that:

$$\exists N_1 \geqslant 0, \exists K_1, K_2 > 0, \forall z \in F, \text{Im}(z) \geqslant N_1 \Rightarrow K_1 \leqslant \exp\left(\alpha y\tan^{-1}\left(\frac{x}{y}\right)\right) \leqslant K_2$$

Now by knowing that q is a conductor of L we have that $\exists d \in \mathbb{N}\setminus\{0\}$, $\exists c \in \mathbb{N}^E$, $\exists \gamma$ a gamma factor of degree d, $\forall a \in E$, $\forall z \in \mathbb{C}$, $q(a,z) = c(a)\,\mathrm{cond}(\gamma)(a,z)$.

Let $a \in E$, we know that by the definition of $\mathrm{cond}(\gamma)$, $\exists a \in \{z \in \mathbb{C} \mid \ \mathrm{Re}(z) > -1\}^{[\![1;d]\!]}$, $\forall z \in \mathbb{C}$, $\mathrm{cond}(\gamma)(a,z) = \prod_{n=1}^{d} (|z + a(n)| + 3)$. Thus:

$$\forall z \in \mathbb{C}, q(a,z) = c(a) \prod_{n=1}^{d} (|z + a(n)| + 3)$$

And we have that:

$$\exists K_3 \geqslant 0, \forall y \in \mathbb{R}, |L(a, 1 + \eta + it)| \leqslant K_3 = K_3$$
$$\exists K_4 \geqslant 0, \forall y \in \mathbb{R}, |L(a, -\eta + it)| \leqslant K_4 q(a, -\eta + it)^{\frac{1}{2}+\eta}$$

We may consider the following function:

$$\begin{array}{rccl} h: & \mathbb{C}\setminus]-\infty;0] & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & L(a,z)g(z)^d \end{array}$$

Let $y \geqslant N_1$ we then have:

$$\left|L(a, 1+\eta+iy)g(1+\eta+iy)^d\right| \leqslant K_3 K_2^d |z|^{-d(\beta - \alpha(1+\eta))}$$

And we have $\beta - \alpha(1+\eta) = \frac{(1+\eta)\left(\frac{1}{2}+\eta\right)}{1+2\eta} - \frac{\frac{1}{2}+\eta}{1+2\eta}(1+\eta) = 0$, which is why we obtain:

$$\left|L(a, 1+\eta+iy)g(1+\eta+iy)^d\right| \leqslant K_3 K_2^d$$

And on the other hand we have:

$$\left|L(a, -\eta+iy)g(-\eta+iy)^d\right| \leqslant K_4 q(a, -\eta+it)^{\frac{1}{2}+\eta} K_2^d |z|^{-d(\beta+\alpha\eta)}$$

And we may calculate $\beta + \alpha\eta = \frac{(1+\eta)\left(\frac{1}{2}+\eta\right)}{1+2\eta} + \frac{\frac{1}{2}+\eta}{1+2\eta}\eta = \frac{\left(\frac{1}{2}+\eta\right)(1+2\eta)}{1+2\eta} = \frac{1}{2} + \eta$ which gives us:

$$\begin{aligned} \left|L(a, -\eta+iy)g(-\eta+iy)^d\right| &\leqslant K_4 K_2^d q(a, -\eta+it)^{\frac{1}{2}+\eta} |z|^{-d\left(\frac{1}{2}+\eta\right)} \\ &= K_4 K_2^d c(a) \prod_{n=1}^{d} \left((|z+a(n)|+3)^{\frac{1}{2}+\eta} |z|^{-\left(\frac{1}{2}+\eta\right)}\right) \end{aligned}$$

By noticing that $\lim_{|y|\to+\infty} \prod_{n=1}^{d} \left((|z+a(n)|+3)^{\frac{1}{2}+\eta} |z|^{-\left(\frac{1}{2}+\eta\right)}\right) = 1$ and by defining $K_5 = c(a)K_4K_2^d$ we have that:

$$\exists K_6 \geqslant 0, \exists N_2 > 0, \forall y \geqslant N_2, \left|L(a, -\eta+iy)g(-\eta+iy)^d\right| \leqslant K_6$$

Let us define $N_3 = \max(N_1, N_2)$ and $X = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in [-\eta; 1+\eta] \text{ and } \operatorname{Re}(z) \geqslant N_3\}$ and since $[-\eta + iN_3; 1+\eta+iN_3]$ is compact we may also define $K_7 = \max(h([-\eta+iN_3; 1+\eta+iN_3]))$. And by defining $K_8 = \max(K_6, K_7)$ we have that:

$$\forall z \in \partial X, |h(z)| \leqslant K_8$$

Since L is an L-function we also know that $\exists A, B \geqslant 0$, $\exists \mu \in [0; 1[$, $\forall x + iy \in \mathbb{C}$, $|f(x+iy)| \leqslant A \exp(B \exp(\eta \frac{\pi}{1+2\eta}) y)$ thus we may apply the previous lemma to $\frac{h}{K_8}$ and get that $\forall z \in X, |h(z)| \leqslant K_8$. Which gives us:

$$\forall z = x+iy \in X, |L(a,z)| \leqslant K_8 \frac{1}{|g(z)|^d} \leqslant \frac{K_8}{K_1} |z|^{d(\beta - \alpha x)}$$

And this concludes the proof.

□

As an immediate consequence of this theorem we can deduce that $\zeta\left(\frac{1}{2}+iy\right) \ll |y|^{\frac{1}{4}+\varepsilon}$. However it is conjectured that $\left(\frac{1}{2}+iy\right) \ll |y|^{\varepsilon}$, and this conjecture can be generalized to L-functions.

Conjecture 2.18 *(Lindelöf's hypothesis)* **:**

Let E be a set, let L be an L-function and let $q = \operatorname{cond}(L)$ then one has that:

$$\forall a \in E, \forall \varepsilon > 0, \left|L\left(a, \frac{1}{2}+iy\right)\right| \ll q\left(a, \frac{1}{2}+iy\right)^{\varepsilon}$$

3 Quantum unique ergodicity

In this section for a Maass cusp form u we shall consider the measure $u^2\mu$ and for an Eisenstein series E we shall consider the measure $|E|^2\mu$. For a hyperbolic disk $D(z,r)$ we wish to establish the behavior of $u^2\mu(D(z,r))$ and $|E|^2\mu(D(z,r))$ as the eigenvalues increase and the radius r decreases. To do so we will need some kind of link between the behavior of the eigenvalues and the behavior in r. So we will consider a function $a(t)$ where t is the eigenvalue and we shall consider the radius $\frac{1}{a(t)}$. We will also need to approximate the function $\mathbb{1}_{D(z,r)}$ by functions of $\mathcal{C}_c^\infty(\mathcal{H})$ to use results that are true for smooth functions. And replacing $\frac{1}{a(t)}$ by ϕ leads to studying $\langle u^2, \phi\rangle$ and $\left\langle |E|^2, \phi\right\rangle$.

We also need the functions ϕ to have a nice behavior which is why we introduce the following definition.

Definition 3.1:

Let $\phi \in \mathcal{C}_c^\infty(\mathcal{H})$ and $a \in [0;+\infty[$, we say that ϕ is a-regular if we have:

$$\forall n \in \mathbb{N}, \exists C \in [0;+\infty[, \|\Delta^n \phi\|_1 \leqslant C a^{2n}$$

3.1 QUE for Maass wave forms

In this section we will first consider a Maass wave form u and try to find an asymptotic expression for the complex measure $u^2\mu$ of a hyperbolic disc. To do so we will need to calculate an asymptotic expression of $\langle u^2, \phi\rangle$. So we can consider a spectral decomposition of Δ and then use the Plancherel formula.

Definition 3.2:

We shall that $\mathcal{B}$ is a spectral basis if it is a orthogonal set of Maass wave cusp forms that satisfies the conclusion of the spectral theorem. For $v \in \mathcal{B}$ we define $\lambda(v)$ such that $\Delta v = \left(\frac{1}{4} + \lambda(v)^2\right)$, we also define the following sets:

- $B(x) = \{v \in \mathcal{B} \mid \lambda(v) \geqslant x\}$
- $A(x,y) = \{v \in \mathcal{B} \mid x \leqslant \lambda(v) \leqslant y\}$

Lemma 3.3 :

Let $\mathcal{B}$ be a spectral basis, let u be a Maass cusp form and $\phi \in \mathcal{C}_c^\infty(\mathcal{H})$. If we consider n to be the counting measure a such $\mathcal{B}$ we obtain the formula:

$$\langle u^2, \phi \rangle = \left\langle u^2, \frac{3}{\pi} \right\rangle \langle 1, \phi \rangle + \int_{\mathcal{B}} \langle u^2, v \rangle \langle v, \phi \rangle \, dn(v) + \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle u^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \left\langle E\left(\cdot, \frac{1}{2} + it\right), \phi \right\rangle d\lambda(t)$$

Proof :

By definition of $\mathcal{B}$ we have:

$$\phi = \left\langle \phi, \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \right\rangle + \int_{\mathcal{B}} \langle \phi, v \rangle \, v dn(v) + \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle \phi, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle E\left(\cdot, \frac{1}{2} + it\right) d\lambda(t)$$

$$u^2 = \left\langle u^2, \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \right\rangle + \int_{\mathcal{B}} \langle u^2, v \rangle \, v dn(v) + \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle u^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle E\left(\cdot, \frac{1}{2} + it\right) d\lambda(t)$$

Then by using the orthogonality of $\mathcal{B}$ and the Eisenstein series we obtain the Plancherel formula:

$$\langle u^2, \phi \rangle = \left\langle u^2, \frac{3}{\pi} \right\rangle \langle 1, \phi \rangle + \int_{\mathcal{B}} \langle u^2, v \rangle \langle v, \phi \rangle \, dn(v) + \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle u^2, E(\cdot, \frac{1}{2} + it) \right\rangle \left\langle E(\cdot, \frac{1}{2} + it), \phi \right\rangle d\lambda(t)$$

□

Now we shall try to find asymptotic expression for each of the terms in the previous lemma by starting with $\int_{\mathcal{B}} \langle u^2, v \rangle \langle v, \phi \rangle \, dn(v)$. And by looking at the eigenvalues of elements of the basis on which we integrate we have have 3 cases: when the eigenvalues are large enough, when they are smaller than at^ε where a is some constant and t the eigenvalue of u and the last case is the rest.

When the eigenvalues are large enough we will use Stirling's formula to show that the integral over those eigenvectors is small. And when the eigenvalues are smaller than at^ε we will need to use Lindelöf's hypothesis to get a nice behavior. Finally we will show that the integration over the rest is small because of a regularity hypothesis over ϕ. All of this will be done in the next 3 lemmas.

Lemma 3.4:

Let $\mathcal{B}$ be a spectral basis. Let u be a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ then $\forall p>0, \exists M>0, \forall C \geqslant M, \exists N>0, \exists K>0, \forall t \geqslant N, \forall \phi \in \mathcal{C}_c^\infty(\mathcal{H})$,

$$\int_{B(2t+C\ln(t))} \left|\left\langle u^2, v\right\rangle \langle v, \phi\rangle\right| dn(v) \leqslant K\,\|\phi\|_1\, t^{-p}$$

Proof :

We must first get an explicit formula for $|\langle u^2, v\rangle \langle v, \phi\rangle|$, to do so we can start using Watson' s formula:

$$\left|\left\langle u^2, v\right\rangle\right| = \frac{\pi}{8} \frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|} \frac{L\left(v,\frac{1}{2}\right) L\left(v,\frac{1}{2}-2it\right)}{L(\operatorname{sym}^2 u, 1)^2 L(\operatorname{sym}^2 v, 1)}$$

By using Stirling's formula (see Appendix) we have that:

$$\frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|}$$

$$\underset{t\to+\infty}{\sim}$$

$$\left|t-\frac{\lambda(v)}{2}\right|^{-\frac{1}{4}}\left|t+\frac{\lambda(v)}{2}\right|^{-\frac{1}{4}}\left|\frac{\lambda(v)}{2}\right|^{-\frac{1}{2}} \exp\left(-\frac{\pi}{2}\left(\left|t+\frac{\lambda(v)}{2}\right|+\left|t-\frac{\lambda(v)}{2}\right|-2t\right)\right)$$

And if $v \in A_C$ we have $\lambda(v) \geqslant 2t + C\ln(t) \geqslant 2t$ which leads us to this simplified expression:

$$\frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|}$$

$$\underset{t\to+\infty}{\sim}$$

$$\left(\frac{\lambda(v)}{2}-t\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}\exp\left(-\frac{\pi}{2}\left(\lambda(v)-2t\right)\right)$$

$$\leqslant$$

$$\left(\frac{\lambda(v)}{2}-t\right)^{-\frac{1}{4}}\exp\left(-\frac{\pi}{2}\left(\lambda(v)-2t\right)\right)$$

Be using the convexity bounds of L-functions we are able to find the following estimates:

$$\begin{aligned}
&\forall\varepsilon>0, L\left(v,\frac{1}{2}\right)\ll\lambda(v)^{\frac{1}{2}+\varepsilon}\leqslant\lambda(v)\\
&\forall\varepsilon>0, L\left(v,\frac{1}{2}-2it\right)\ll(\lambda(v)^2t)^{\frac{1}{4}+\varepsilon}\leqslant\lambda(v)^2t\\
&\forall\varepsilon>0, L(\operatorname{sym}^2 u,1)\geqslant t^{-\varepsilon}\geqslant t^{-1}\\
&\forall\varepsilon>0, L(\operatorname{sym}^2 v,1)\geqslant\lambda(v)^{-\varepsilon}\geqslant\lambda(v)^{-1}
\end{aligned}$$

And this gives us:

$$\frac{L\left(u\times u\times v,\frac{1}{2}\right)}{L(\operatorname{sym}^2 u,1)^2L(\operatorname{sym}^2 v,1)}\ll\frac{\lambda(v)\lambda(v)^2t}{t^{-2}\lambda(v)^{-1}}=\lambda(v)^4t^3$$

Using this we are able to get an upper asymptotic bound for $|\langle u^2,v\rangle|$. And for $|\langle v,\phi\rangle|$ we have the following bound which is proved in [6]:

$$|\langle v,\phi\rangle|\ll\|\phi\|_1\left(\frac{1}{4}+\lambda(v)^2\right)^{\frac{1}{4}}\leqslant\|\phi\|_1\left(2\lambda(v)^2\right)^{\frac{1}{4}}\leqslant\|\phi\|_1 2\lambda(v)^2$$

Using this we finally get and define ψ by:

$$\left|\left\langle u^2,v\right\rangle\langle v,\phi\rangle\right|\ll\|\phi\|_1\lambda(v)^6t^3\left(\frac{\lambda(v)}{2}-t\right)^{-\frac{1}{4}}\exp\left(-\frac{\pi}{2}\left(\lambda(v)-2t\right)\right)=\psi(\lambda(v))$$

For what follows let us define the following function:

$$\begin{array}{rccl} g: & B(2t+C\ln(t)) & \longrightarrow & [2t+C\ln(t);+\infty[\\ & v & \longmapsto & \lambda(t)\end{array}$$

And consider the measure $m = n_g$ which counts the eigenvalues and is defined by $n_g(E) = n(g^{-1}(E))$ by change of variables one has $\int\limits_{B(2t+C\ln(t))} \psi(g(v))dn(v) = \int\limits_{[2t+C\ln(t);+\infty[} \psi(x)dm(x)$.
And we may once again apply a change of variables using:

$$\begin{array}{rccc} h: & [2t + C\ln(t); +\infty[& \longrightarrow & [0; +\infty[\\ & x & \longmapsto & x - 2t - C\ln(t) \end{array}$$

And by considering m_h to be the image measure this gives us that:

$$\begin{aligned} \int\limits_{[2t+C\ln(t);+\infty[} \psi(x)dm(x) &= \int\limits_{[2t+C\ln(t);+\infty[} \psi(h(x) + 2t + C\ln(t))dm(x) \\ &= \int\limits_{[0;+\infty[} \psi(x + 2t + C\ln(t))dm_h(x) \end{aligned}$$

Before estimating this integral we may first find an easy to compute an upper bound:

$$\begin{aligned} &\psi(x + 2t + C\ln(t)) \\ &= \|\phi\|_1 (x + 2t + C\ln(t))^6 t^3 \left(\frac{x + 2t + C\ln(t)}{2} - t\right)^{-\frac{1}{4}} \exp\left(-\frac{\pi}{2}(x + 2t + C\ln(t) - 2t)\right) \\ &= \|\phi\|_1 (x + 2t + C\ln(t))^6 t^3 \left(\frac{x + C\ln(t)}{2}\right)^{-\frac{1}{4}} \exp\left(-\frac{\pi}{2}(x + C\ln(t))\right) \\ &\leqslant \|\phi\|_1 (x + 2t + C\ln(t))^6 t^3 \exp\left(-\frac{\pi}{2}(x + C\ln(t))\right) \\ &= t^{-\frac{\pi}{2}C+3} \|\phi\|_1 (x + 2t + C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) \end{aligned}$$

This gives us:

$$\int\limits_{[0;+\infty[} \psi(x+2t+C\ln(t))dm_h(x) \leqslant t^{-\frac{\pi}{2}C+3} \|\phi\|_1 \int\limits_{[0;+\infty[} (x+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) dm_h(x)$$

Thus by noticing that $[0;+\infty[=\bigcup_{n\in\mathbb{N}}[n;n+1[$ we may compute:

$$\begin{aligned}
&\int_{[0;+\infty[} (x+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) dm_h(x) \\
&= \sum_{n=0}^{+\infty} \int_{[n;n+1[} (x+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) dm_h(x) \\
&\leqslant \sum_{n=0}^{+\infty} \int_{[n;n+1[} (n+1+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}n\right) dm_h(x) \\
&= \sum_{n=0}^{+\infty} (n+1+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}n\right) m_h([n;n+1[) \\
&\leqslant \sum_{n=0}^{+\infty} (n+1+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}n\right) m_h([0;n+1])
\end{aligned}$$

To estimate the measure one can use Weyl's law that states $\exists K>0, m([0,n])\leqslant Kn^2$. Thus we have that:

$$\begin{aligned}
m_h([0;n+1]) &= m(h^{-1}([0;n+1)) \\
&= m([2t+C\ln(t);n+1+2t+C\ln(t)]) \\
&\leqslant m([0;n+1+2t+C\ln(t)]) \\
&\leqslant K(n+1+2t+C\ln(t))^2
\end{aligned}$$

Thus we have:

$$\begin{aligned}
&\int_{[0;+\infty[} (x+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) dm_h(x) \\
&\leqslant K\sum_{n=0}^{+\infty} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right)
\end{aligned}$$

Since there is an exponential, this sum converges strongly thus we can easily find an explicit upper bound. Indeed we have:

$$\begin{aligned}
&\sum_{n=0}^{+\infty} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right) \\
&= \sum_{n=0}^{2t+C\ln(t)} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right) \\
&+ \sum_{n=2t+C\ln(t)}^{+\infty} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right)
\end{aligned}$$

So on one hand we have:

$$\begin{aligned}
&\sum_{n=0}^{2t+C\ln(t)} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right) \\
&\leqslant \sum_{n=0}^{2t+C\ln(t)} (n+1+2t+C\ln(t))^8 \\
&\leqslant \sum_{n=0}^{2t+C\ln(t)} 2^8(1+2t+C\ln(t))^8 \\
&= 2^8(1+2t+C\ln(t))^8 \sum_{n=0}^{2t+C\ln(t)} 1 \\
&= 2^8(1+2t+C\ln(t))^8(2t+C\ln(t)+1) \\
&= 2^8(1+2t+C\ln(t))^9
\end{aligned}$$

And one the other hand we have:

$$\begin{aligned}
&\sum_{n=2t+C\ln(t)}^{+\infty} (n+1+2t+C\ln(t))^8 \exp\left(-\frac{\pi}{2}n\right) \\
&\leqslant \sum_{n=2t+C\ln(t)}^{+\infty} (2(n+1))^8 \exp\left(-\frac{\pi}{2}n\right) \\
&\leqslant 2^8 \sum_{n=0}^{+\infty} (n+1)^8 \exp\left(-\frac{\pi}{2}n\right) = 2^8 S
\end{aligned}$$

Where S is a constant that is independent of t. So we have that:

$$\begin{aligned}
&\int_{[0;+\infty[} (x+2t+C\ln(t))^6 \exp\left(-\frac{\pi}{2}x\right) dm_h(x) \\
&\leqslant 2^8((1+2t+C\ln(t))^9 + S)
\end{aligned}$$

Which yields:

$$\int_{B(2t+C\ln(t))} \psi(g(v))dn(v) \leqslant t^{-\frac{\pi}{2}C+3} \|\phi\|_1 K2^8((1+2t+C\ln(t))^9 + S)$$

And we obtain:

$$\int_{B(2t+C\ln(t))} \left|\langle u^2, v\rangle \langle v, \phi\rangle\right| dn(v) \leqslant t^{-\frac{\pi}{2}C+3} \|\phi\|_1 K2^8((1+2t+C\ln(t))^9 + S)$$

Thus by choosing C to be large enough we obtain the wanted result.

□

Lemma 3.5 :

Let $\mathcal{B}$ be a spectral basis. Let u be a Maass wave cusp form of eigenvalue $\frac{1}{4}+t^2$. Then:

$$\forall p>0, \forall \varepsilon>0, \forall C>0, \exists N>0, \exists K>0, \forall \phi \in \mathcal{C}_c^{\infty}(\mathcal{H}), \forall a>0,$$
$$t \geqslant N \text{ and } \phi \text{ is } a\text{-regular} \Rightarrow \int_{A(at^{\varepsilon}, 2t+C\ln(t))} \left|\left\langle u^2, v\right\rangle \langle v, \phi\rangle\right| dn(v) \leqslant K \|\phi\|_1 t^{-p}$$

Proof :

Once again we may make use of Watson's formula (see [7]):

$$\left|\left\langle u^2, v\right\rangle\right| = \frac{\pi}{8} \frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|} \frac{L\left(v, \frac{1}{2}\right) L\left(v, \frac{1}{2}-2it\right)}{L(\operatorname{sym}^2 u, 1)^2 L(\operatorname{sym}^2 v, 1)}$$

And we may also use Stirling's formula to obtain that $\exists C \geqslant 0$ such that:

$$\frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|}$$
$$\leqslant$$
$$C\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(t+\frac{\lambda(v)}{2}\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}$$
$$\times \exp\left(-\frac{\pi}{2}\left(\left|t+\frac{\lambda(v)}{2}\right|+\left|t-\frac{\lambda(v)}{2}\right|-2t\right)\right)$$

Thus for $\lambda(v) \leqslant t$ we have:

$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(t+\frac{\lambda(v)}{2}\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}$$
$$\times \exp\left(-\frac{\pi}{2}\left(\left|t+\frac{\lambda(v)}{2}\right|+\left|t-\frac{\lambda(v)}{2}\right|-2t\right)\right)$$
$$=$$
$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(t+\frac{\lambda(v)}{2}\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}$$

And for $\lambda(v) > t$ we have:

$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(t+\frac{\lambda(v)}{2}\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}$$
$$\times\exp\left(-\frac{\pi}{2}\left(\left|t+\frac{\lambda(v)}{2}\right|+\left|t-\frac{\lambda(v)}{2}\right|-2t\right)\right)$$
$$=$$
$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}\exp\left(-\frac{\pi}{2}\left(\lambda(v)-2t\right)\right)$$
$$\leqslant$$
$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}}\left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}}\left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}}$$

For dealing with L functions we may use this estimate that we have established previously:

$$\frac{L\left(v,\frac{1}{2}\right)L\left(v,\frac{1}{2}-2it\right)}{L(\operatorname{sym}^2 u,1)^2L(\operatorname{sym}^2 v,1)} \ll \frac{\lambda(v)\lambda(v)^2t}{t^{-2}\lambda(v)^{-1}} = \lambda(v)^4t^3$$

All the previous estimates will help us in dealing with $|\langle u^2, v\rangle|$. For dealing with $|\langle v,\phi\rangle|$ we will use the fact that ϕ is a-regular. Indeed for Maass form $v \in \mathcal{B}$ one has $\Delta v = (\frac{1}{4}+\lambda(v)^2)v$, thus we have:

$$\begin{aligned}\left(\frac{1}{4^2}+\lambda(v)^2\right)^k\langle v,\phi\rangle &= \left\langle \Delta^k v,\phi\right\rangle \\ &= \left\langle v,\Delta^k\phi\right\rangle\end{aligned}$$

Now we have:

$$\begin{aligned}\left|\left\langle v,\Delta^k\phi\right\rangle\right| &\leqslant \|v\|_\infty \left\|\Delta^k\phi\right\|_1 \\ &\leqslant \|v\|_\infty Ca^{2k}\end{aligned}$$

For dealing with $\|v\|_\infty$ we may once again use the bound $\|v\|_\infty \ll \left(\frac{1}{4}+\lambda(v)^2\right)^{\frac{1}{4}}$ to obtain:

$$|\langle v,\phi\rangle| \ll \left(\frac{1}{4^2}+\lambda(v)^2\right)^{\frac{1}{4}}\left(\frac{a^2}{\frac{1}{4}+\lambda(v)^2}\right)^k$$

And if $\lambda(v) \geqslant at^{\varepsilon}$ we have $\left(\frac{a^2}{\frac{1}{4}+\lambda(v)^2}\right)^k \leqslant t^{-2\varepsilon n}$, thus we have $|\langle v,\phi\rangle| \ll \lambda(v)^{\frac{1}{2}}t^{-2\varepsilon k}$. And we may estimate for $b \geqslant 0$:

$$\begin{aligned}
&\int_{A(at^\varepsilon,b)} \left|\left\langle u^2, v\right\rangle \langle v,\phi\rangle\right| dn(v) \\
&\ll \int_{A(at^\varepsilon,b)} \left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}} \left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}} \left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}} \lambda(v)^{\frac{1}{2}} t^{-2\varepsilon k} dn(v) \\
&\leqslant \int_{A(at^\varepsilon,b)} \left(\frac{1}{4}\right)^{-\frac{1}{4}} \left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}} \left(\frac{\lambda(v)}{2}\right)^{-\frac{1}{2}} b^{\frac{1}{2}} t^{-2\varepsilon k} dn(v) \\
&\leqslant \int_{A(at^\varepsilon,b)} b^{\frac{1}{2}} t^{-2\varepsilon k} dn(v) = b^{\frac{1}{2}} t^{-2\varepsilon k} n(A(at^\varepsilon, b)) \\
&\leqslant b^{\frac{1}{2}} t^{-2\varepsilon k} n(A(0,b))
\end{aligned}$$

Thus replacing b by $2t + C\ln(t)$ we obtain:

$$\int_{A(at^\varepsilon,2t+C\ln(t))} \left|\left\langle u^2, v\right\rangle \langle v,\phi\rangle\right| dn(v) \ll (2t+C\ln(t))^{\frac{1}{2}} t^{-2\varepsilon k} n(A(0,2t+C\ln(t)))$$

And by Weyl's law we also have that $n(A(0,2t+C\ln(t))) \ll (2t+C\ln(t))^2$, which yields:

$$\begin{aligned}
\int_{A(at^\varepsilon,2t+C\ln(t))} \left|\left\langle u^2, v\right\rangle \langle v,\phi\rangle\right| dn(v) &\ll (2t+C\ln(t))^{\frac{1}{2}} t^{-2\varepsilon k} (2t+C\ln(t))^2 \\
&\leqslant t^{-2\varepsilon k} (2t+C\ln(t))^3
\end{aligned}$$

And by choosing k to be large enough we obtain the wanted result.

□

Lemma 3.6 :

Let us consider that Lindelöf's hypothesis as true. Let $\mathcal{B}$ be a spectral basis. Let u be a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$, then $\forall\delta>0, \forall\varepsilon>0, \exists N>0, \exists K>0, \forall t \geqslant N, \forall\phi \in \mathcal{C}_c^\infty(\mathcal{H}), \forall a \geqslant 0$

$$\varepsilon < \delta \text{ and } a \leqslant t^{1-\delta} \text{ and } \phi \text{ is } a\text{-regular}$$
$$\Rightarrow$$
$$\int_{A(0,at^\varepsilon)} \left|\left\langle u^2, v\right\rangle \langle v,\phi\rangle\right| dn(v) \leqslant K \|\phi\|_2 \left(t^{-\frac{1}{2}+\varepsilon} + a^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon}\right)$$

Proof :

Let $\varepsilon \in]0;\delta[$.
Consider $\eta > 0$ to be such that $\eta \leqslant \varepsilon$ and $\frac{\varepsilon(1+\eta)+2\eta}{2} \leqslant \varepsilon$. By Lindelöf's hypothesis for L-functions let us consider $N, K_1 > 0$ to be such that:

$$\lambda(v), t \geqslant N \Rightarrow \frac{L\left(v, \frac{1}{2}\right) L\left(v, \frac{1}{2} - 2it\right)}{L(\operatorname{sym}^2 u, 1)^2 L(\operatorname{sym}^2 v, 1)} \leqslant K_1 \lambda(v)^{\frac{\eta}{2}} t^{\frac{\eta}{2}}$$

In the previous estimate we were able to get rid of the symmetric square L-functions thanks to their nice behavior, for more information the reader can see [2]. By defining the bilinear product $\langle f, g\rangle_{A(0,at^\varepsilon)} = \int\limits_{A(0,at^\varepsilon)} f(v)\overline{g(v)} dn(v)$ we may use Cauchy Schwarz and Bessel's inequality to obtain the following upper bound:

$$\int\limits_{A(0,at^\varepsilon)} \left|\left\langle u^2, v\right\rangle \langle v, \phi\rangle\right| dn(v) = \left\langle \left|\left\langle u^2, v\right\rangle\right|, \left|\langle v, \phi\rangle\right| \right\rangle_{A(0,at^\varepsilon)}$$

$$\leqslant \left(\int\limits_{A(0,at^\varepsilon)} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \right)^{\frac{1}{2}} \left(\int\limits_{A(0,at^\varepsilon)} \left|\langle v, \phi\rangle\right|^2 dn(v) \right)^{\frac{1}{2}}$$

$$\leqslant \left(\int\limits_{A(0,at^\varepsilon)} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \right)^{\frac{1}{2}} \|\phi\|_2$$

All the expressions above are finite because $A(0, at^\varepsilon)$ is a finite set, thus its measure is finite. Now what we need to do is find a nice upper bound for $\int\limits_{A(0,at^\varepsilon)} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v)$. Of course we can use once again Stirling's formula to get $\exists K_2 \geqslant 0$,

$$\frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right| \left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right| \left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2 \left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|}$$

$$\leqslant$$

$$K_2 \left(\frac{1}{4^2} + \left(t - \frac{\lambda(v)}{2}\right)^2 \right)^{-\frac{1}{8}} \left(\frac{\lambda(v)}{2} + t \right)^{-\frac{1}{4}} \left(\frac{1}{4^2} + \left(\frac{\lambda(v)}{2}\right)^2 \right)^{-\frac{1}{4}}$$

We can simplify this even more by noticing that we have that $\left(\frac{\lambda(v)}{2}+t\right)^{-\frac{1}{4}} \leqslant t^{-\frac{1}{4}}$ and:

$$\left(\frac{1}{4^2}+\left(t-\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{8}} \leqslant \left(t-\frac{\lambda(v)}{2}\right)^{-\frac{1}{4}}$$
$$= t^{-\frac{1}{4}}\left(1-\frac{\lambda(v)}{2t}\right)^{-\frac{1}{4}}$$

And since we have that $\lambda(v) \leqslant at^{\varepsilon}$ and that $a \leqslant t^{1-\delta}$ we have $\lambda(v) \leqslant t^{1-\delta+\varepsilon}$. Which implies that $\frac{\lambda(v)}{t} \leqslant \frac{t^{1-\delta+\varepsilon}}{2t} = \frac{t^{-\delta+\varepsilon}}{2}$ where $-\delta+\varepsilon < 0$. Thus we may consider that $\exists K_3 \geqslant 0, \forall t \geqslant N$,

$$t^{-\frac{1}{4}}\left(1-\frac{\lambda(v)}{2t}\right)^{-\frac{1}{4}} \leqslant K_3 t^{-\frac{1}{4}}$$

Which leads us to:

$$\frac{\left|\Gamma\left(\frac{\frac{1}{2}+2it+i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+2it-i\lambda(v)}{2}\right)\right|\left|\Gamma\left(\frac{\frac{1}{2}+i\lambda(v)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1+2it}{2}\right)\right|^2\left|\Gamma\left(\frac{1+2i\lambda(v)}{2}\right)\right|}$$
$$\leqslant K_2 K_3 t^{-\frac{1}{2}}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{4}}$$

And we have that for $\lambda(v) \geqslant N$:

$$\left|\langle u^2, v\rangle\right| \leqslant \frac{\pi}{8}K_1K_2K_3 t^{-\frac{1}{2}}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{4}}\lambda(v)^{\frac{\eta}{2}}t^{\frac{\eta}{2}}$$

And be defining $K_4 = (\frac{\pi}{8}K_1K_2K_3)^2$ we have:

$$\left|\langle u^2, v\rangle\right|^2 \leqslant K_4 t^{-1}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}}\lambda(v)^{\eta}t^{\eta}$$
$$= K_4 t^{-1+\eta}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}}\lambda(v)^{\eta}$$

Using this upper bound we may start by computing:

$$\int\limits_{A(N,at^{\varepsilon})}\left|\langle u^2, v\rangle\right|^2 dn(v) \leqslant K_4 \int\limits_{A(N,at^{\eta})} t^{-1+\varepsilon}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}}\lambda(v)^{\eta}dn(v)$$
$$= K_4 t^{-1+\eta}\int\limits_{A(N,at^{\varepsilon})}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}}\lambda(v)^{\eta}dn(v)$$

By defining $M = \max(\mathbb{N}\cap[N; at^\varepsilon])$ we can then write $A(N, at^\varepsilon) = \bigcup_{k=N}^{M} A(N, at^\varepsilon)\cap A(k, k+1)$, which will then give us:

$$\begin{aligned}
&\int_{A(N,at^\varepsilon)} \left(\frac{1}{4^2} + \left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}} \lambda(v)^\eta dn(v) \\
&= \sum_{k=N}^{M} \int_{A(N,at^\varepsilon)\cap A(k,k+1)} \left(\frac{1}{4^2} + \left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}} \lambda(v)^\eta dn(v) \\
&\leqslant \sum_{k=N}^{M} \int_{A(N,at^\varepsilon)\cap A(k,k+1)} \left(\frac{\lambda(v)^2}{4}\right)^{-\frac{1}{2}} \lambda(v)^\eta dn(v) \\
&\leqslant 2\sum_{k=N}^{M} \int_{A(N,at^\varepsilon)\cap A(k,k+1)} \lambda(v)^{-1+\eta} dn(v)
\end{aligned}$$

To calculate this expression we may also use Weyl's bound that says that for $k \geqslant N$ we have $\exists K_5 > 0, n(A(k, k+1)) \leqslant K_4 k$.

$$\begin{aligned}
\sum_{k=N}^{M} \int_{A(N,at^\varepsilon)\cap A(k,k+1)} \lambda(v)^{-1+\eta} dn(v) &\leqslant \sum_{k=N}^{M} \int_{A(N,at^\varepsilon)\cap A(k,k+1)} k^{-1+\eta} dn(v) \\
&= \sum_{k=N}^{M} k^{-1+\eta} n(A(N, at^\varepsilon) \cap A(k, k+1)) \\
&\leqslant \sum_{k=N}^{M} k^{-1+\eta} n(A(k, k+1)) \\
&\leqslant \sum_{k=1}^{M} k^{-1+\eta} K_5 k
\end{aligned}$$

And we have that:

$$\begin{aligned}
\sum_{k=1}^{N} k^{-1+\eta} K_5 k = K_5 \sum_{k=1}^{N} k^\eta &\leqslant K_4 \sum_{k=1}^{N} N^\eta \\
&\leqslant K_5 N^\eta \sum_{k=1}^{N} 1 = K_4 N^\eta N \\
&= K_5 N^{\eta+1} \leqslant K_5 (at^\varepsilon)^{\eta+1} \\
&= K_5 a^{1+\eta} t^{\varepsilon(1+\eta)}
\end{aligned}$$

So we finally obtain that:

$$\int_{A(N,at^{\varepsilon})} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \leqslant 2K_4K_5t^{-1+\eta}a^{1+\eta}t^{\varepsilon(1+\eta)}$$

$$= 2K_4K_5t^{-1+\varepsilon(1+\eta)+\eta}a^{1+\eta}$$

And we have that $a \leqslant t^{1-\delta} \leqslant t \Rightarrow a^{\eta} \leqslant t^{\eta}$ which leads us to:

$$\int_{A(N,at^{\varepsilon})} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \leqslant 2K_4K_5t^{-1+\varepsilon(1+\eta)+2\eta}a$$

On the other hand if $\lambda(v) < N$ we have a finite number of such values thus we are able to handle the behavior of L-functions for small values and we have that there is $K_6 \geqslant 0$ such that:

$$\left|\left\langle u^2, v\right\rangle\right|^2 \leqslant K_6t^{-1+\eta}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}}$$

Thus:

$$\int_{A(0,N)} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \leqslant \int_{A(0,N)} K_6t^{-1+\eta}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}} dn(v)$$

$$= K_6t^{-1+\eta}\int_{A(0,N)}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}} dn(v)$$

And by defining $K_7 = \displaystyle\int_{A(0,N)}\left(\frac{1}{4^2}+\left(\frac{\lambda(v)}{2}\right)^2\right)^{-\frac{1}{2}} dn(v)$ we have that:

$$\int_{A(0,N)} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \leqslant K_6K_7t^{-1+\eta}$$

Thus by defining $K_8 = \max(2K_4K_5, K_6K_7)$ we have:

$$\int_{A(0,at^{\varepsilon})} \left|\left\langle u^2, v\right\rangle\right|^2 dn(v) \leqslant K_8t^{-1+\eta} + K_8at^{-1+\varepsilon(1+\eta)+2\eta}$$

We then obtain:

$$\int_{A(0,at^{\varepsilon})} \left|\left\langle u^2, v\right\rangle\left\langle v, \phi\right\rangle\right| dn(v) \leqslant \left\|\phi\right\|_2\left(K_8t^{-1} + K_8at^{-1+\varepsilon(1+\eta)+2\eta}\right)^{\frac{1}{2}}$$

$$= K_8^{\frac{1}{2}}\left\|\phi\right\|_2\left(t^{-1+\eta} + at^{-1+\varepsilon(1+\eta)+2\eta}\right)^{\frac{1}{2}}$$

To conclude one can notice that for $a, b \in [0;+\infty[$ we have that $(a^2+b^2)^{\frac{1}{2}} \leqslant a+b$ thus for $c, d \in [0;+\infty[$ we have that $(c+d)^{\frac{1}{2}} \leqslant c^{\frac{1}{2}} + d^{\frac{1}{2}}$ which leads to:

$$\int\limits_{A(0,at^{\varepsilon})} \left|\left\langle u^2, v\right\rangle \langle v, \phi\rangle\right| dn(v) \leqslant K_8^{\frac{1}{2}} \|\phi\|_2 \left(t^{-1+\eta} + at^{-1+\varepsilon(1+\eta)+2\eta}\right)^{\frac{1}{2}}$$

$$\leqslant K_8^{\frac{1}{2}} \|\phi\|_2 \left(t^{-\frac{1}{2}+\frac{\eta}{2}} + a^{\frac{1}{2}} t^{-\frac{1}{2}+\frac{\varepsilon(1+\eta)+2\eta}{2}}\right)$$

$$\leqslant K_8^{\frac{1}{2}+\frac{\eta}{2}} \|\phi\|_2 \left(t^{-\frac{1}{2}+\varepsilon} + a^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon}\right)$$

□

Now let us take a look at the contribution of $\int\limits_{\mathbb{R}} \left\langle u^2, E(\cdot, \frac{1}{2}+il)\right\rangle \left\langle E(\cdot, \frac{1}{2}+il), \phi\right\rangle d\lambda(l)$.

We shall deal with this term in the same kind way, by separating this integral over 3 parts. However to do we need the following result which be used in the same way as we used Watson's formula.

Lemma 3.7:

Let u be a Maass form of eigenvalue $\frac{1}{4}+t^2$ and let us define $Z(u,s) = \frac{\zeta(s)}{\zeta(2s)} L(\operatorname{sym}^2 u, s)$ then we have that:

$$\left\langle u^2, E(\cdot, s)\right\rangle = \frac{\pi^{-\overline{s}}}{8} \frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)} Z(\overline{s}, u)$$

Proof :

Recall that $E(z,s) = \sum\limits_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s$ thus we may use the unfolding technique to obtain what follows:

$$\begin{aligned}
\left\langle u^2, E(\cdot,s)\right\rangle &= \int\limits_{\mathcal{F}} u^2(z)E(z,\overline{s})d\mu(z) = \int\limits_{\mathcal{F}} u^2(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^{\overline{s}}d\mu(z) \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int\limits_{\mathcal{F}} u^2(z)\mathrm{Im}(\gamma z)^{\overline{s}}d\mu(z) \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int\limits_{\gamma\mathcal{F}} u^2(z)\mathrm{Im}(z)^{\overline{s}}d\mu(z) \\
&= \int\limits_{\Gamma_\infty\backslash\mathcal{H}} u^2(z)\mathrm{Im}(z)^{\overline{s}}d\mu(z) \\
&= \int\limits_{[0,1]\times[0;+\infty[} u^2(x,y)y^{\overline{s}}d\left(\frac{\lambda^2}{y^2}\right)(x,y) \\
&= \int\limits_{[0,1]\times[0;+\infty[} u^2(x,y)y^{\overline{s}-2}d\lambda^2(x,y)
\end{aligned}$$

Now once can remember that since u is a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ we can use it's Fourrier expansion:

$$u(x+iy) = y^{\frac{1}{2}}\sum_{n=1}^{+\infty}a_n\,\mathrm{K}_{it}(2\pi ny)\cos(2\pi nx)$$

Thus we have:

$$u^2(x+iy) = y\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}a_na_m\,\mathrm{K}_{it}(2\pi ny)\,\mathrm{K}_{it}(2\pi my)\cos(2\pi nx)\cos(2\pi mx)$$

Using this expression we can compute the previous integral:

$$\begin{aligned}
&\left\langle u^2, E(\cdot,s)\right\rangle = \int\limits_{[0,1]\times[0;+\infty[} u^2(x,y)y^{\overline{s}-2}d\lambda^2(x,y) \\
&= \int\limits_{[0,1]\times[0;+\infty[} y\left(\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}a_na_m\,\mathrm{K}_{it}(2\pi ny)\,\mathrm{K}_{it}(2\pi my)\cos(2\pi nx)\cos(2\pi mx)\right)y^{\overline{s}-2}d\lambda^2(x,y) \\
&= \int\limits_{[0;+\infty[}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}a_na_m\,\mathrm{K}_{it}(2\pi ny)\,\mathrm{K}_{it}(2\pi my)\left(\int\limits_{[0,1]}\cos(2\pi nx)\cos(2\pi mx)d\lambda(x)\right)y^{\overline{s}-1}d\lambda(y)
\end{aligned}$$

One can also notice that by a simple calculation we have:

$$\int_{[0,1]} \cos(2\pi nx)\cos(2\pi mx)d\lambda(x) = \begin{cases} 1 \text{ if } n = m \\ 0 \text{ otherwise} \end{cases}$$

This leads us to the following computation:

$$\begin{aligned}
&\left\langle u^2, E(\cdot, s)\right\rangle \\
&= \int_{[0;+\infty[} \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} a_n a_m \,\mathrm{K}_{it}(2\pi ny)\,\mathrm{K}_{it}(2\pi my)\left(\int_{[0,1]} \cos(2\pi nx)\cos(2\pi mx)d\lambda(x)\right) y^{\overline{s}-1}d\lambda(y) \\
&= \int_{[0;+\infty[} \sum_{n=1}^{+\infty} a_n^2 \,\mathrm{K}_{it}(2\pi ny)^2 y^{s-1}d\lambda(y) \\
&= \sum_{n=1}^{+\infty} a_n^2 \int_{[0;+\infty[} \mathrm{K}_{it}(2\pi ny)^2 y^{s-1}d\lambda(y)
\end{aligned}$$

And we also have that:

$$\int_{[0;+\infty[} \mathrm{K}_{it}(2\pi ny)^2 y^{\overline{s}-1}d\lambda(y) = \frac{\pi^{-\overline{s}}}{8}\frac{1}{n^{\overline{s}}}\frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)}$$

So by defining $Z(s,u)$ by the following expression:

$$Z(s,u) = \sum_{n=1}^{+\infty} a_n^2 = \frac{\zeta(s)}{\zeta(2s)}L(\mathrm{sym}^2\, u, s)$$

We are able to continue the calculations to find an explicit formula:

$$\begin{aligned}
\left\langle u^2, E(\cdot, s)\right\rangle &= \sum_{n=1}^{+\infty} a_n^2 \int_{[0;+\infty[} \mathrm{K}_{it}(2\pi ny)^2 y^{s-1}d\lambda(y) \\
&= \sum_{n=1}^{+\infty} a_n^2 \frac{\pi^{-\overline{s}}}{8}\frac{1}{n^{\overline{s}}}\frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)} \\
&= \frac{\pi^{-\overline{s}}}{8}\frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)}\sum_{n=1}^{+\infty}\frac{a_n^2}{n^{\overline{s}}} \\
&= \frac{\pi^{-\overline{s}}}{8}\frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)} Z(\overline{s}, u)
\end{aligned}$$

And this concludes the proof.

□

Lemma 3.8:

Let u be a Maass form of eigenvalue $\frac{1}{4}+t^2$ then we have that:

$$\exists K \geqslant 0, \forall t \geqslant 0, \forall l \in \mathbb{R},$$
$$\left|\frac{\pi^{-\overline{s}}}{8}\frac{\Gamma\left(\frac{\overline{s}+2it}{2}\right)\Gamma\left(\frac{\overline{s}}{2}\right)^2\Gamma\left(\frac{\overline{s}-2it}{2}\right)}{\Gamma\left(\overline{s}\right)}\right|$$
$$\leqslant$$
$$Kt^{-\frac{1}{4}}(1+(2t-l)^2)^{-\frac{1}{8}}(1+l^2)^{-\frac{1}{4}}\exp\left(-\frac{\pi}{2}\left(\left|t-\frac{l}{2}\right|+\left|t+\frac{l}{2}\right|\right)\right)$$

Lemma 3.9:

Let u be a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ then we have that:

$$\forall p \geqslant 0, \exists c \geqslant 0, \exists K \geqslant 0, \exists N \geqslant 0, \forall t \geqslant N,$$
$$\int_{[2t+c\ln(t);+\infty[}\left|\left\langle u^2, E(\cdot,\frac{1}{2}+il)\right\rangle\left\langle E(\cdot,\frac{1}{2}+il),\phi\right\rangle\right|d\lambda(l) \leqslant K\left\|\phi\right\|_1 t^{-p}$$

Lemma 3.10:

Let u be a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ then:

$$\forall p \geqslant 0, \forall \varepsilon > 0, \forall C \geqslant 0, \exists K, \exists N \geqslant 0, \geqslant 0, \forall a \geqslant 0, \forall \phi \in \mathcal{C}_c^\infty(\mathcal{H}),$$
$$t \geqslant N \text{ and } \phi \text{ is } a\text{-regular}$$
$$\Rightarrow$$
$$\int_{[at^\varepsilon;2t+c\ln(t)]}\left|\left\langle u^2, E(\cdot,\frac{1}{2}+il)\right\rangle\left\langle E(\cdot,\frac{1}{2}+il),\phi\right\rangle\right|d\lambda(l) \leqslant Kt^{-p}$$

Lemma 3.11 :

Let u be a Maass cusp form of eigenvalue $\frac{1}{4}+t^2$ then:

$$\forall \varepsilon > 0, \exists K \geqslant 0,$$
$$\int\limits_{[0;at^{\varepsilon}]} \left| \left\langle u^2, E(\cdot, \frac{1}{2}+il) \right\rangle \left\langle E(\cdot, \frac{1}{2}+il), \phi \right\rangle \right| d\lambda(l) \leqslant K \exp\left(-\frac{\pi}{2}t\right)$$

By using all the previous lemmas one can easily deduce the following proposition which is one of the most important results of this section.

Proposition 3.12 :

Let u be a Maass cusp form of eigen value $\frac{1}{4}+t^2$ then we have that:

$$\forall p > 0, \forall \delta > 0, \forall \varepsilon \in]0; \delta[, \exists K \geqslant 0, \exists N \geqslant 0, \forall \phi \in \mathcal{C}_c^{\infty}(\mathcal{H}), \forall a \geqslant 0,$$
$$t \geqslant N \text{ and } a \leqslant t^{1-\delta} \text{ and } \phi \text{ is } a\text{-regular}$$
$$\Rightarrow$$
$$\left| \left\langle u^2, \phi \right\rangle - \left\langle u^2, \frac{3}{\pi} \right\rangle \langle 1, \phi \rangle \right| \leqslant K \left(\|\phi\|_2 t^{-\frac{1}{2}+\varepsilon} \left(1 + a^{\frac{1}{2}}\right) + \|\phi\|_1 t^{-p} \right)$$

Using the previous proposition we are ready to prove unique quantum ergodicity for a shrinking set of discs. But before doing so we need the following technical lemma that expresses that we can use functions $\phi \in \mathcal{C}_c^{\infty}(\mathcal{H})$ obtain a similar proposition for the characteristic function of a disc.

Corollary 3.13 :

Let u be a Maass cusp form of eigen value $\frac{1}{4}+t^2$ then we have that:

$$\forall p > 0, \forall \delta > 0, \forall \varepsilon \in]0; \delta[, \exists K \geqslant 0, \exists N \geqslant 0, \forall z \in \mathcal{H}, \forall r > 0, \forall a \geqslant 0,$$
$$t \geqslant N \text{ and } a \leqslant t^{1-\delta}$$
$$\Rightarrow$$
$$\left| \left\langle u^2, \mathbb{1}_{D(z,r)} \right\rangle - \left\langle u^2, \frac{3}{\pi} \right\rangle \left\langle 1, \mathbb{1}_{D(z,r)} \right\rangle \right| \leqslant K \left(\mu(D(z,r))^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon} \left(1 + a^{\frac{1}{2}}\right) + \mu(D(z,r)) t^{-p} \right)$$

Proof :

Let $z \in \mathcal{H}$, let $r > 0$, let $\varepsilon > 0$ and let $M = \sup\limits_{z \in D(z,r)} |u(z)|^2$.
Let $\eta > 0$.
Consider $\phi \in \mathcal{C}_c^\infty(\mathcal{H})$ such that $\forall w \in D(z,r)$, $\phi(w) = 1$ and such that $\mu(\text{supp}(\phi)\backslash D(z,r)) < \eta$ and such that ϕ is a-regular. Then we have that:

$$\begin{aligned}
&\left|\left\langle u^2, \mathbb{1}_{D(z,r)}\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \mathbb{1}_{D(z,r)}\right\rangle\right| \\
&\leqslant \left|\left\langle u^2, \mathbb{1}_{D(z,r)}\right\rangle - \left\langle u^2, \phi\right\rangle\right| + \left|\left\langle u^2, \phi\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \langle 1, \phi\rangle\right| \\
&+ \left|\left\langle u^2, \frac{3}{\pi}\right\rangle \langle 1, \phi\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \mathbb{1}_{D(z,r)}\right\rangle\right| \\
&= \left|\left\langle u^2, \mathbb{1}_{D(z,r)} - \phi\right\rangle\right| + \left|\left\langle u^2, \phi\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \langle 1, \phi\rangle\right| + \left|\left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \phi - \mathbb{1}_{D(z,r)}\right\rangle\right|
\end{aligned}$$

Let us start by showing that the first and last terms are small. To do so we can notice that:

$$\begin{aligned}
\left\|\mathbb{1}_{D(z,r)} - \phi\right\|_1 &= \int\limits_{\mathcal{H}} \left|\mathbb{1}_{D(z,r)}(z) - \phi(z)\right| d\mu(z) \\
&= \int\limits_{\mathcal{H}} \mathbb{1}_{\text{supp}(\phi)\backslash D(z,r)}(z) \, |\phi(z)| \, d\mu(z) \\
&\leqslant \int\limits_{\mathcal{H}} \mathbb{1}_{\text{supp}(\phi)\backslash D(z,r)}(z) d\mu(z) \\
&= \mu(\text{supp}(\phi)\backslash D(z,r)) < \eta
\end{aligned}$$

Thus we can calculate:

$$\begin{aligned}
\left|\left\langle u^2, \mathbb{1}_{D(z,r)} - \phi\right\rangle\right| &= \left|\int\limits_{\mathcal{H}} u(z)^2 \left(\mathbb{1}_{D(z,r)}(z) - \phi(z)\right) d\mu(z)\right| \\
&\leqslant \int\limits_{\mathcal{H}} |u(z)|^2 \left|\mathbb{1}_{D(z,r)}(z) - \phi(z)\right| d\mu(z) \\
&\leqslant M \int\limits_{\mathcal{H}} \left|\mathbb{1}_{D(z,r)}(z) - \phi(z)\right| d\mu(z) \\
&= M \left\|\mathbb{1}_{D(z,r)} - \phi\right\|_1 < M\eta
\end{aligned}$$

And similarly we have that:

$$\left|\left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \phi - \mathbb{1}_{D(z,r)}\right\rangle\right| < \left|\left\langle u^2, \frac{3}{\pi}\right\rangle\right| \eta$$

And we can also use the previous lemma to have that:

$$\left|\left\langle u^2, \phi\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \langle 1, \phi\rangle\right| \leqslant K\left(\|\phi\|_2 t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \|\phi\|_1 t^{-p}\right)$$

Then we can apply the triangle inequalities to have $\|\phi\|_1 \leqslant \left\|\phi - \mathbb{1}_{D(z,r)}\right\|_1 + \left\|\mathbb{1}_{D(z,r)}\right\|_1 < \left\|\mathbb{1}_{D(z,r)}\right\|_1 + \eta$. As well as $\left\|\phi - \mathbb{1}_{D(z,r)}\right\|_2 + \left\|\mathbb{1}_{D(z,r)}\right\|_2 < \left\|\mathbb{1}_{D(z,r)}\right\|_2 + \eta^{\frac{1}{2}}$. So we have that:

$$\begin{aligned}
&\left|\left\langle u^2, \phi\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \langle 1, \phi\rangle\right| \\
&\leqslant K\left(\left\|\mathbb{1}_{D(z,r)}\right\|_2 t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \left\|\mathbb{1}_{D(z,r)}\right\|_1 t^{-p}\right) \\
&+ K\left(\eta^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \eta t^{-p}\right)
\end{aligned}$$

Now by putting all of the pieces together we get:

$$\begin{aligned}
&\left|\left\langle u^2, \mathbb{1}_{D(z,r)}\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \mathbb{1}_{D(z,r)}\right\rangle\right| \\
&\leqslant K\left(\left\|\mathbb{1}_{D(z,r)}\right\|_2 t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \left\|\mathbb{1}_{D(z,r)}\right\|_1 t^{-p}\right) \\
&+ K\left(\eta^{\frac{1}{2}} t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \eta t^{-p}\right) + M\eta + \left|\left\langle u^2, \frac{3}{\pi}\right\rangle\right| \eta
\end{aligned}$$

Since this is true $\forall \eta > 0$ we can take the limit of η approaching 0 to obtain:

$$\left|\left\langle u^2, \mathbb{1}_{D(z,r)}\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \mathbb{1}_{D(z,r)}\right\rangle\right| \leqslant K\left(\left\|\mathbb{1}_{D(z,r)}\right\|_2 t^{-\frac{1}{2}+\varepsilon}\left(1 + a^{\frac{1}{2}}\right) + \left\|\mathbb{1}_{D(z,r)}\right\|_1 t^{-p}\right)$$

This concludes the proof.

□

Theorem 3.14 *(QUE for Maass forms)* **:**

Let us define the set of Maass cusp forms $A = \left\{u \in \mathbb{C}^{\mathcal{H}} \mid u \text{ is a Maass cusp form}\right\}$ and $B = \left\{t \in \mathbb{R} \mid \exists u \in A, -\Delta u = \left(\frac{1}{4} + t^2\right) u\right\}$. Let $\delta > 0$ and let $a \in]0; +\infty[^B$ such that $\lim\limits_{t \to +\infty} a(t) = +\infty$ and $\forall t \in B$, $a(t) \leqslant t^{\frac{1}{3}-\delta}$. Let $f \in A^B$ such that $\forall t \in A$, $-\Delta f(t) = \left(\frac{1}{4} + t^2\right) f(t)$ and $\left\langle f(t)^2, \frac{3}{\pi}\right\rangle = 1$ then we have that:

$$\left(f(t)^2 \mu\right)\left(D\left(z, \frac{1}{a(t)}\right)\right) \underset{t \to \infty}{\sim} \mu\left(D\left(z, \frac{1}{a(t)}\right)\right)$$

Proof :

Let $\delta > 0$ then by the previous corollary we have:

$$\begin{gathered}\exists K_1 \geqslant 0, \exists N_1 \geqslant 0, \forall z \in \mathcal{H}, \forall r > 0, \forall b \geqslant 0,\\ t \geqslant N_1 \text{ and } b \leqslant t^{1-\delta}\\ \Rightarrow\\ \left|\left\langle u^2, \mathbb{1}_{D(z,r)}\right\rangle - \left\langle u^2, \frac{3}{\pi}\right\rangle \left\langle 1, \mathbb{1}_{D(z,r)}\right\rangle\right| \leqslant K_1 \left(\mu(D(z,r))^{\frac{1}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} \left(1 + b^{\frac{1}{2}}\right) + \mu(D(z,r)) t^{-1}\right)\end{gathered}$$

And since we have that $\lim\limits_{t\to+\infty} a(t) = +\infty$ we have that $\lim\limits_{t\to+\infty} \frac{1}{a(t)} = 0$ thus we may use the surface area formula for the hyperbolic disc to obtain that:

$$\exists N_2 \geqslant 0, \exists K_2, K_3 \geqslant 0, \forall t \geqslant N_2, K_3 a(t)^{-2} \leqslant \mu\left(D\left(z, \frac{1}{a(t)}\right)\right) \leqslant K_2 a(t)^{-2}$$

And we may consider $N_3 = \max(N_1, N_2)$ and for $t \geqslant N_3$ we have that:

$$\begin{aligned}&\left|\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle - \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle\right|\\ &\leqslant K_1 \left(\mu\left(D\left(z, \frac{1}{a(t)}\right)\right)^{\frac{1}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} \left(1 + a(t)^{\frac{1}{2}}\right) + \mu\left(D\left(z, \frac{1}{a(t)}\right)\right) t^{-1}\right)\\ &\leqslant K_1 \left(K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} \left(1 + a(t)^{\frac{1}{2}}\right) + K_2 a(t)^{-2} t^{-1}\right)\\ &= K_1 \left(K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} a(t)^{\frac{1}{2}} + K_2 a(t)^{-2} t^{-1}\right)\end{aligned}$$

And we have that:

$$\begin{aligned}\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle &= \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle + \left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle - \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle\\ &= \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle \left(1 + \frac{\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle - \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle}{\left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle}\right)\end{aligned}$$

By our previous calculations we have that for $t \geqslant N_3$:

$$\begin{aligned}
&\left|\frac{\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle - \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle}{\left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle}\right| \\
&\leqslant K_1 \frac{\left(K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} a(t)^{\frac{1}{2}} + K_2 a(t)^{-2} t^{-1}\right)}{\left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle} \\
&\leqslant \frac{K_1}{K_3} a(t)^2 \left(K_2^{\frac{1}{2}} a(t)^{-1} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} a(t)^{-\frac{1}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2 a(t)^{-2} t^{-1}\right) \\
&= \leqslant \frac{K_1}{K_3} \left(K_2^{\frac{1}{2}} a(t) t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} a(t)^{\frac{3}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2 t^{-1}\right)
\end{aligned}$$

And since $\forall t \in B, a(t) \leqslant t^{\frac{1}{3}-\delta}$ we have that:

$$\begin{aligned}
K_2^{\frac{1}{2}} a(t) t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} a(t)^{\frac{3}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} a(t)^{\frac{1}{2}} &\leqslant K_2^{\frac{1}{2}} t^{\frac{1}{3}-\delta} t^{-\frac{1}{2}+\frac{\delta}{2}} + K_2^{\frac{1}{2}} \left(t^{\frac{1}{3}-\delta}\right)^{\frac{3}{2}} t^{-\frac{1}{2}+\frac{\delta}{2}} \\
&= K_2^{\frac{1}{2}} t^{-\frac{1}{6}-\frac{\delta}{2}} + K_2^{\frac{1}{2}} t^{\frac{1}{2}-\frac{3}{2}\delta} t^{-\frac{1}{2}+\frac{\delta}{2}} \\
&= K_2^{\frac{1}{2}} t^{-\frac{1}{6}-\frac{\delta}{2}} + K_2^{\frac{1}{2}} t^{-\delta}
\end{aligned}$$

Thus this proves that:

$$\lim_{t \to +\infty} \frac{\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle - \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle}{\left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle} = 0$$

And this fact gives us the wanted result:

$$\left\langle f(t)^2, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle \underset{t \to +\infty}{\sim} \left\langle 1, \mathbb{1}_{D\left(z,\frac{1}{a(t)}\right)}\right\rangle$$

□

3.2 QUE for Eisenstein series

In this section we shall have a heuristic approach for showing quantum unique ergodicity for Eisenstein series.

We wish to apply the technique of the previous section to the measure $\left|E\left(\cdot, \frac{1}{2}+it\right)\right|^2 \mu$ however we cannot apply Parceval's formula because $\left|E\left(\cdot, \frac{1}{2}+it\right)\right|^2$ grows too fast. To be able to manage this problem be can introduced regularized integrals which consists of substracting problematic terms, for more information one can see [9]. And this yields a regularized version of the Parseval's formula.

Theorem 3.15 :

Let F be a family of functions of compact support contained in a fixed set and let $a(t) \ll t^{1-\delta}$. Let us also suppose that there is a sequence of numbers c such that every $\phi \in F$ satisfies $\left\|\Delta^k \phi\right\| \leqslant c(k)a^{2k}$ for any k. Then we have that for any $p \geqslant 0$:

$$\left\langle \left| E\left(\cdot, \frac{1}{2} + it\right)\right|^2, \phi \right\rangle$$
$$= \ln\left(\frac{1}{4} + t^2\right)\left\langle \phi, \frac{3}{\pi}\right\rangle + O\left(a^{\frac{1}{2}} \left\|\phi\right\|_2 t^{-\frac{1}{6}+\varepsilon}\right) + O\left(\frac{\ln(t)}{\ln(\ln(t))}\left\|\phi\right\|_1\right) + O(t^{-p})$$

The constants only depend on ε and the sequence c.

We may use the same kind of arguments as in the section 4.1 to obtain the following theorem which states that we have the quantum unique ergodicity for Eisenstein series.

Theorem 3.16** (QUE for Eisenstein series) **:

Let $\delta > 0$ and let a be a function from $\mathbb{R}$ to $]0; +\infty[$ such that $\lim_{t\to+\infty} a(t) = +\infty$ and $\forall t \in B$, $a(t) \leqslant t^{\frac{1}{3}-\delta}$. Then we have that:

$$\left(\left|E\left(\cdot, \frac{1}{2} + it\right)\right|^2 \mu\right)\left(D\left(z, \frac{1}{a(t)}\right)\right) \underset{t\to\infty}{\sim} \ln\left(\frac{1}{4} + t^2\right)\mu\left(D\left(z, \frac{1}{a(t)}\right)\right)$$

Appendix: The gamma function

Definition 3.17:

We can then define the gamma function as follows:

$$\begin{array}{rccc} \Gamma: & \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \displaystyle\int_0^{+\infty} \exp(-t) t^{z-1} dt \end{array}$$

Theorem 3.18** (Stirling's formula) **:

Let us consider $f(z) = \displaystyle\int_0^{+\infty} \frac{x - \lfloor x \rfloor - (x - \lfloor x \rfloor)^2}{2(z+x)^2} dx$ which converges uniformly in every compact subset of $\mathbb{C} \backslash]-\infty; 0]$. Then what has that:

$$\forall z \in \mathbb{C} \backslash]-\infty; 0], \gamma(z) = (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} \exp(-z) \exp(f(z))$$

This formula has many corollaries but in this document we shall only use the 2 following results.

Corollary 3.19:

Let $x \in \mathbb{R}$ then we have that:

$$\Gamma(x+iy) \underset{|y| \to +\infty}{\sim} (2\pi)^{\frac{1}{2}} |y|^{\frac{1}{2}-x} \exp\left(-\frac{\pi}{2}|y|\right)$$

Corollary 3.20:

For the Γ function we have, $\forall x \in]0; +\infty[, \exists m, M > 0, \forall y \in \mathbb{R}$,

$$m \left(x^2+y^2\right)^{\frac{1}{2}\left(x-\frac{1}{2}\right)} \exp\left(-\frac{\pi}{2}|y|\right) \leqslant |\Gamma(x+iy)| \leqslant M \left(x^2+y^2\right)^{\frac{1}{2}\left(x-\frac{1}{2}\right)} \exp\left(-\frac{\pi}{2}|y|\right)$$

Proof :

This result comes from the fact that $\frac{|\Gamma(x+iy)|}{(x^2+y^2)^{\frac{1}{2}\left(x-\frac{1}{2}\right)}\exp\left(-\frac{\pi}{2}|y|\right)}$ is continuous for a fixed x and is converging to 1 as $|y|$ gets large. And since Γ has no zeros this function has no zeros as well which explains the lower bound.

$\square$

4 Conclusion

We were able to prove the quantum unique conjecture for decreasing hyperbolic discs using analytical tools such as L-functions. However we supposed that Lindelöf's hypothesis is true which is still unproved and it seems that this hypothesis is necessary for our theorems to be true.

Thanks to this project I was able to learn a lot about analytical number theory. Many concepts were new to me such as Maass forms and general L-functions. To be honest there are still many things that I do not fully understand but at least I was able to see how these objects can be used and why it is interesting to study them.

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