Sum of coefficients of Dirichlet series

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Abstract

The aim of this article is to give a generalization of a result by Chandrasekharan and Narasimhan. The Perron Formula reformulates us the sum of the coefficients of a Dirichlet series as an integral. This integral can then be shifted to capture poles, and then shifted again by a functional equation. We shall see which estimate we can give for this integral, and how does it apply to the case of L-functions.

Contents

1 Introduction

1.1 Motivation

Given a sequence $(a_n)_{n\in\mathbb{N}}$, we can consider the associated power series $\sum_{n\geqslant 0} a_n z^n$. This series, which converges inside a certain disk (eventually $\{0\}$), defines an holomorphic function. Conversely, every holomorphic function can locally be written as a power series. If $\sum_{n\geqslant 0} a_n z^n$ and $\sum_{n\geqslant 0} b_n z^n$ are two power series, then their product is given by

$$\sum_{n\geqslant 0} c_n z^n, \text{ where } c_n = \sum_{k+l=n} a_k b_l. \tag{1}$$

The relation defining c_n is a convolution product, that could be described as additive since the sum runs over the couples (k,l) satisfying $k+l=n$.
Now we give an example of the multiplicative version of this convolution product.

Example 1. *Let Λ be the Von Mangoldt function:*

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^\alpha \text{ for some } \alpha \in \mathbb{N}^* \\ 0 & \text{otherwise} \end{cases}.$$

Λ satisfies

$$\sum_{dl=n} \Lambda(d) = \log(n) \text{ , for all } n \geqslant 1.$$

This sum is different than the sum defining c_n in (1). The corresponding generating function is given by considering the Dirichlet series $F(s) = \sum_{n\geqslant 1} \frac{\Lambda(n)}{n^s}$. *We can show that*

$$F(s) = -\frac{\zeta'(s)}{\zeta(s)}, \text{ for } \Re(s) > 1.$$

We have built a bridge between an arithmetical object and an analytic one. Problems associated to Λ can now be translated into analytic problems. For instance, it can be shown that the prime number theorem is equivalent to

$$\sum_{n<x} \Lambda(n) \sim x.$$

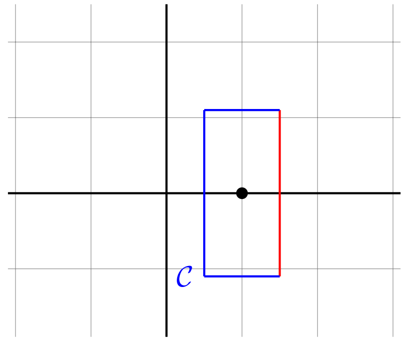
We will see that the Perron formula enable us to write this sum as the following integral:

$$\sum_{n<x} \Lambda(n) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'(w)}{\zeta(w)w} x^w w dw, \tag{2}$$

for $c > 1$. We can then give an estimate for this integral by using estimations of the ζ function. We can start by, instead of integrating over the whole vertical line, integrate over a segment and estimate the rest.

$$\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(w)}{\zeta(w)w} x^w dw = \int_{c-iT}^{c+iT} \frac{\zeta'(w)}{\zeta(w)w} x^w dw + R,$$

for some $T > 0$ that we might choose depending on x. The new integral can be evaluated by closing the line of integration to capture the pole at $s = 1$ of the integrand.



$$\int_{c-iT}^{c+iT} \frac{\zeta'(w)}{\zeta(w)w} x^w dw = x + R',$$

because x is the residue of the integrand at $s = 1$. In [Mur08], the error term $R + R'$ is shown to be a $o(x)$. This is the idea of the proof of the prime number theorem.
However, the error term obtained using this method is not optimal. It is not easy to see because the prime number theorem is not a trivial result. We can follow the same idea to evaluate the trivial sum $\sum_{n<x} 1$. We will see that the error term we obtain in this case is $\mathcal{O}\left(x^{3/4+\epsilon}\right)$, which is a little disappointing. This will motivate us to find a new way to evaluate the integral in the Perron formula.

1.2 Properties of Dirichlet series

Power series and Dirichlet series have similar propreties, so we shall first give the results for power series without proving them, then we will give the equivalent results for Dirichlet series.

Definition 1.1 (Power Series)**.** If $(a_n)_{n\in\mathbb{N}}$ is any sequence of complex numbers, we define the formal *associated power series* by

$$\textstyle\sum_{n\geqslant 0} a_n z^n.$$

Definition 1.2 (Dirichlet Series)**.** If $(a_n)_{n\in\mathbb{N}}$ is any sequence of complex numbers, we define the formal *associated Dirichlet series* by

$$\textstyle\sum_{n\geqslant 0} a_n n^{-s}.$$

And we shall define both series as functions once we establish some convergence theorem.

Proposition 1. *Let $f(z) = \sum_{n\geqslant 0} a_n z^n$ be a power series. There exist a radius $R \geqslant 0$ such that $f(z)$ converges for all z such that $|z| < R$ and for no z such that $|z| > R$. This radius R is called* radius of convergence*.*

Proposition 2. *Let $f(s) = \sum_{n\geqslant 0} a_n n^{-s}$ be a Dirichlet series. There exist an abscissa $\sigma_c \in \overline{\mathbb{R}}$ such that $f(s)$ converges for all s such that $\Re(s) > \sigma_c$ and for no s such that $\Re(s) < \sigma_c$. This abscissa σ_c is called* abscissa of convergence*.*

Remark. This abscissa of convergence might be equal to $-\infty$, in which case the Dirichlet series converges for all $s \in \mathbb{C}$. Furthermore, the abscissa of convergence might be equal to $+\infty$, in which case the Dirichlet series converges for no $s \in \mathbb{C}$.

Since we will often need to consider the real and the imaginary part of some complex number s, we shall use the following notation from now on: $s = \sigma + it$ and $s_c = \sigma_c + it_c$.

Proof. (Proposition 2) If there is no s such that $f(s)$ converges, then there is nothing to prove. Hence, we can suppose the existence of $s_0 \in \mathbb{C}$ such that $f(s_0)$ converges.
Let $R(k) = \sum_{n>k} a_n n^{-s_0}$ be the rest of the series for every integer k, and consider two integer N, M and $s \in \mathbb{C}$ such that $\Re(s) > \Re(s_0)$. We can then apply the *Abel transform*

$$\begin{aligned}
\sum_{n=M+1}^{N} a_n n^{-s} &= \sum_{n=M+1}^{N} (R(n-1) - R(n)) n^{s_0 - s} \\
&= \sum_{n=M+1}^{N} R(n-1) n^{s_0-s} - \sum_{n=M+1}^{N} R(n) n^{s_0-s} \\
&= R(M)(M+1)^{s_0-s} - R(N)N^{s_0-s} + \sum_{n=M+1}^{N-1} R(n)((n+1)^{s_0-s} - n^{s_0-s}) \\
&= R(M)(M+1)^{s_0-s} - R(N)N^{s_0-s} + \sum_{n=M+1}^{N-1} R(n)(s_0 - s) \int_{n}^{n+1} t^{s_0-s-1} dt \\
&= R(M)(M+1)^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s) \int_{M+1}^{N} R(t) t^{s_0-s-1} dt.
\end{aligned}$$

Let $\epsilon > 0$. We can choose M so that $R(k) < \epsilon$ for all $k \geqslant M$:

$$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leqslant 2\epsilon + |s_0 - s|\epsilon \int_{M+1}^{N} t^{\Re(s_0-s-1)} dt \tag{3}$$

$$\leqslant 2\epsilon + |s_0 - s|\epsilon \int_{M+1}^{\infty} t^{\Re(s_0-s-1)} dt \tag{4}$$

$$\leqslant 2\epsilon + |s_0 - s|\epsilon \left[\frac{t^{\Re(s_0-s)}}{\Re(s_0 - s)} \right]_{M+1}^{\infty} \tag{5}$$

$$\leqslant 2\epsilon + |s_0 - s|\epsilon (0 - \frac{(M+1)^{\Re(s_0-s)}}{\Re(s_0 - s)}) \tag{6}$$

$$\leqslant 2\epsilon + \frac{|s_0 - s|\epsilon}{\Re(s - s_0)}. \tag{7}$$

So the Dirichlet Series converge for all s such that $\Re(s) > \Re(s_0)$.
If we now consider the infimum of the real parts of those s_0, and note it σ_c, we have shown that the Dirichlet series converge for all $\sigma > \sigma_c$ and for no $\sigma < \sigma_c$. $\square$

Definition 1.3. For a Dirichlet series $f(s) = \sum_{n \geqslant 1} a_n n^{-s}$ with abscissa of convergence σ_c, we can also consider the abscissa of convergence of the series $\sum_{n \geqslant 1} |a_n| n^{-s}$. This is called the *abscissa of absolute convergence* of f.

Remark. If a_n is positive for all n, then $\sigma_c = \sigma_a$ by definition, but in general this might not be true. However, we always have $\sigma_a > \sigma_c$.

Example 2. *The Zeta function* $\sum_{n \geqslant 1} 1/n^s$ *has abscissa of convergence* 1*, by the Riemann criterion. Its abscissa of absolute convergence is also* 1*. Now we will take a look at an example where* $\sigma_c \neq \sigma_a$*. Consider* $\sum_{n \geqslant 1} (-1)^n / n^s$*. Once again, we have in this case* $\sigma_a = 1$*. However,* *because the sequence $(-1)^n n^{-s}$ is alterning, the series converges as soon as $\sigma > 0$. Hence $\sigma_c = 0$. This is actually the case were σ_c and σ_a are as far as possible, because we always have $\sigma_a \leqslant \sigma_c + 1$. [MV12]*

The general term of a Dirichlet series is holomorphic in s. When we are on the half plane $\sigma > \sigma_a$, the sum converges uniformly on compacts, so that the Dirichlet series is also holomorphic. We will see that this is still true if we only suppose to be on the half plane $\sigma > \sigma_c$.

Proposition 3. *If R is the radius of convergence of the power series $f(z) = \sum_{n\geqslant 0} a_n z^n$, then $f(z)$ is holomorphic in the open disc $\{z \in \mathbb{C} \big| |z| < R\}$.*

Proposition 4. *If σ_c is the abscissa of conergence of the Dirichlet series $f(s) = \sum_{n\geqslant 0} a_n n^s$, then $f(s)$ is holomorphic in the open half plane $\{s \in \mathbb{C} \big| Re(s) > \sigma_c\}$.*

Proof. Let $E = \{s \in \mathbb{C} \big| \Re(s) > \sigma_c\}$ and for all $K > 1$ we denote $E_K = \{s \in \mathbb{C} \big| \Re(s) > \sigma_c, |s - s_c| \leqslant K(\sigma - \sigma_c)\}$.
Note that we have:

$$E = \bigcup_{K>1} E_K. \tag{8}$$

We will show that the Dirichlet Serie converge uniformly in every E_K. We will continue from (7),

$$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leqslant 2\epsilon + \frac{|s_c - s|\epsilon}{\Re(s - s_c)} \leqslant 2\epsilon + K\epsilon.$$

For fixed K, we can choose ϵ adequately so that $2\epsilon + K\epsilon$ is as small as we want it to be. Hence, we have shown that the series converges uniformly on every set E_k, which implies that $f(s)$ is holomorphic on these sets. Now, since being holomorphic is a local propriety, $f(s)$ must be holomorphic everywhere on E. □

In the introduction, we have seen that the coefficients of the product of two Dirichlet series or two power series are given by a convolution product. More precisely:

Proposition 5. *Let $f(z) = \sum_{n\geqslant 0} a_n z^n$ and $g(z) = \sum_{n0} b_n z^n$ be two power series with radius of convergence being R_1 and R_2.*

Then the power series $h(z) = \sum_{n\geqslant 0} c_n z^n$ with $c_n = \sum_{k+l=n} a_k b_l$ converges for all $z < min(R_1, R_2)$ and then we have $h(z) = f(z)g(z)$.

Proposition 6. *Let $f(s) = \sum_{n\geqslant 1} a_n n^{-s}$ and $g(s) = \sum_{n\geqslant 1} b_n n^{-s}$ be two Dirichlet Series with abscissa of absolute convergence being σ_{a1} and σ_{a2}.*

Then the Dirichlet series $h(s) = \sum_{n\geqslant 1} c_n n^{-s}$ with $c_n = \sum_{kl=n} a_k b_l$ converges for all $\sigma > max(\sigma_{a1}, \sigma_{a2})$ and then we have $h(s) = f(s)g(s)$.

Proof. We can write formally

$$\sum_{k\geqslant 1} a_k k^{-s} \sum_{l\geqslant 1} b_l l^{-s} = \sum_{k\geqslant 1} \sum_{l\geqslant 1} a_k b_l (kl)^{-s} = \sum_{n\geqslant 1} n^{-s} (\sum_{kl=n} a_k b_l).$$

The second equality is justified if the sums converge absolutely, which is the case if we suppose $\sigma > \max(\sigma_{a1}, \sigma_{a2})$. □
We give some examples of arithmetic functions and formulas, and their generating functions.

Example 3. *Let $d(n)$ denote the number of divisors of n. By definition, we have*

$$d(n) = \sum_{d|n} 1.$$

The translation of this formula in the analytic world is

$$\sum_{n \geqslant 1} \frac{d(n)}{n^s} = \zeta(s)^2.$$

Now let $\nu(n)$ be the number of prime factors in n, and let $\mu(n) = (-1)^{\nu(n)}$ is n if square-free, and $\mu(n) = 0$ otherwise. μ is the mobius function, and satisfies

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ otherwise} \end{cases}.$$

The analytic equivalent is

$$\sum_{n \geqslant 1} \frac{\mu(n)}{n^s} = 1/\zeta(s).$$

Recall the Von Mangoldt function Λ from the introduction (1). Since $\log(n) = \sum_{kl=n} \Lambda(k)$ for all $n \in \mathbb{N}^*$, and $\zeta'(s) = \sum_{n>1} \frac{-\log(n)}{n^s}$ for all $\Re(s) > 1$, we have the analytic relation:

$$\sum_{n>1} \frac{\Lambda(n)}{n^s} = -\zeta'(s)/\zeta(s).$$

We will often consider integrals along vertical lines, so its important to have an idea of the behavior of Dirichlet series as t goes to ∞.

Proposition 7. *If f is a Dirichlet series, then for all $0 < \epsilon < \delta < 1$*

$$f(s) = O((|t| + 1)^{1-\delta+\epsilon}) \tag{9}$$

uniformly for all $\sigma > \sigma_c + \delta$.

Proof. Recall from Proposition 2 that for all $\sigma_0 > \sigma_c$ and $N > M$,

$$\sum_{n=M+1}^{N} a_n n^{-s} = R(M)(M+1)^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s) \int_{M+1}^{N} R(t)t^{s_0-s-1}dt.$$

Thus by taking $s_0 = +\sigma_c + \epsilon$ and by letting N go to ∞ we get

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{M} a_n n^{-s} + R(M)(M+1)^{\sigma_c+\epsilon-s} + (\sigma_c + \epsilon - s) \int_{M+1} R(t)t^{\sigma_c+\epsilon-s-1}dt.$$

Since the Dirichlet series $f(\sigma_c + \epsilon)$ converges, its general term vanishes. Hence there exist a uniform constant C so that $a_n < Cn^{\sigma_c+\epsilon}$. Furthermore, the function R vanishes at ∞ so it is bounded by some constant. Hence

$$f(s) \leqslant \mathcal{C}\Big(\sum_{n=1}^{M} n^{-\delta+\epsilon} + M^{-\delta+\epsilon} + \frac{|\sigma_c + \epsilon - s|}{\sigma - \sigma_c - \epsilon} M^{\sigma_c+\epsilon-\sigma}\Big)$$

Moreover, we know that $\sum_{n=1}^{M} n^{-\delta+\epsilon} \leqslant \int_0^M u^{-\delta+\epsilon} du = \mathcal{C} M^{1-\delta+\epsilon}$, which is the dominant term. By choosing $M = |t| + 1$, we get the $\mathcal{O}$ term of the Proposition. □

The most famous Dirichlet series is the Zêta function $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$. We will prove its analytical properties in the next section. The Zêta function is closely connected to the study of prime numbers, since it can be written as the Euler product $\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$. We can give such products for other Dirichlet series:

Proposition 8. *If a_n is totally multiplicative, in the sense that $a_{nm} = a_n a_m$ for all integers n, m, then we have the Euler product:*

$$\sum_{n \geqslant 1} a_n n^{-s} = \prod_{p \text{ prime}} (1 - a_p p^{-s})^{-1}$$

which holds for every s such that $\sigma > \sigma_a$.

2 The Perron formula

In this section we will prove the Perron Formula, which gives us the sum of the coefficients of a Dirichlet series as an integral. We will then apply the formula to evaluate the sum $\sum_{n<x} 1$. Of course, we don't need the Perron formula nor complexe analysis to evaluate this sum. But it is interesting to see what error term the Perron formula gives in this simple example, in order to have an idea of the limitations of this method.

2.1 Perron formula

We have seen that the Dirichlet series enable us to change an arithmetic problem into an analytic one. Reciprocally, every Dirichlet series is associated to a unique sequence of coefficients $(a_n)_{n \in \mathbb{N}}$. But how do we in pratice have access to these coefficients? The Perron formula gives us an answer in the form of an integral.

Theorem 1. *Let $x \in \mathbb{R}^+ \backslash \mathbb{N}$ and $c > 0$. We also suppose that $\sigma > \sigma_c - c$. Then:*

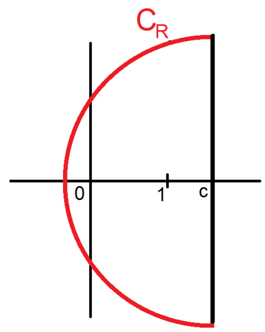
$$\sum_{n<x} a_n n^{-s} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w} dw. \tag{10}$$

The idea of the proof is to write f as its Dirichlet series and then invert the sum and the integral. The obtained sum can then be evaluated by using the next Lemma

Lemma 1. *Let $\alpha > 0$. Then*

$$\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\alpha^w}{w} dw = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha > 1 \end{cases}$$

Proof. Case $\alpha > 1$: In this case, for all $\sigma_w < c$, we have $\sigma_w \log(a) < c \log(a)$ so $a^{\sigma_w} < a^c$. Hence, a^w is bounded on the half plane $\{\sigma_w < c\}$.
Now we denote by C_R the left half-circle centered in c and whose diameter is on $[c - iR, c + iR]$. For R sufficiently large, C_R contains the point 0, which is the only singularity of f.



We can also see that $wf(w) \underset{w \to 0}{\longrightarrow} 1$, so the residue of this pole is 1. By the Residue Theorem, we conclude that

$$\frac{1}{2i\pi}\int_{C_R}\frac{\alpha^w}{w}dw + \frac{1}{2i\pi}\int_{c-iR}^{c+iR}\frac{\alpha^w}{w}dw = 1. \tag{11}$$

On the other hand, we can evaluate integral (11) by using integration by parts. For fixed R, we will integrate α^w and derivate $\frac{1}{w}$.

$$\int_{C_R}\frac{\alpha^w}{w}dw = \left[\frac{a^w}{w\log a}\right]_{c-iR}^{c+iR} + \int_{C_R}\frac{a^w}{w^2\log a}dw. \tag{12}$$

Since a^w is bounded on the half plane $\sigma_w < c$, we can see that

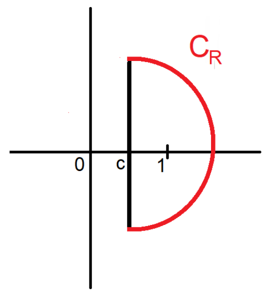
$$\begin{aligned}
\left|\left[\frac{a^w}{w\log a}\right]_{c-iR}^{c+iR}\right| &= \left|\frac{a^{c+iR}}{(c+iR)\log a} - \frac{a^{c-iR}}{(c-iR)\log a}\right| \\
&\le \left|\frac{a^{c+iR}}{(c+iR)\log a}\right| + \left|\frac{a^{c-iR}}{(c-iR)\log a}\right| \\
&= 2\frac{a^c}{\sqrt{c^2+R^2}\log a},
\end{aligned}$$

which goes to 0 as R goes to ∞. Now we show that the integral in (12) also goes to 0.

$$\begin{aligned}
\left|\int_{C_R}\frac{a^w}{w^2\log a}dw\right| &\le \int_{C_R}\frac{a^c}{(c^2+R^2)\log a}dw \\
&\le \pi R\frac{a^c}{(R-c)^2\log a},
\end{aligned}$$

which goes to 0 as R goes to ∞.
In conclusion, we have $\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\alpha^w}{w}dw = 1$.

Case $\alpha < 1$: In this case, for all $\sigma_w > c$, we have $\sigma_w \log(a) < c\log(a)$ so that $a^{\sigma_w} > a^c$. Hence, a^w is bounded on the half plane $\{\sigma_w > c\}$.
Let C_R be the right half circle centered in c and whose diameter is on $[c-iR, c+iR]$. Since $c > 0$, the curve consisting of C_R end its diameter doesn't contain any singularities of the integrant.



Hence, by the Residue Theorem

$$\int_{C_R} \frac{\alpha^w}{w} dw + \int_{c+iR}^{c-iR} \frac{a^w}{w} dw = 0.$$

By a similar argument, we can show that the first term goes to 0 as R goes to ∞.
In conclusion, we have $\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{a^w}{w} dw = 0$ $\square$

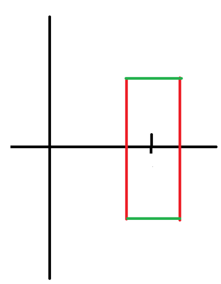
We can now prove the Perron Formula.

Proof. Suppose in the first place that $\sigma_a < \sigma + c$, so that f is absolutely convergent on the vertical line of abscissa $\sigma + c$. Then we can write f as its series and invert the sum and the integral:

$$\frac{1}{2i\pi} \int_{c-\infty}^{c+\infty} f(s+w) \frac{x^w}{w} dw = \frac{1}{2i\pi} \sum_{n \geq 1} a_n \int_{c-\infty}^{c+\infty} n^{-s-w} \frac{x^w}{w} dw \tag{13}$$

$$= \frac{1}{2i\pi} \sum_{n \geq 1} a_n n^{-s} \int_{c-\infty}^{c+\infty} \frac{(\frac{x}{n})^w}{w} dw. \tag{14}$$

We now need to show that the integral in (14) is equal to 1 if $x < n$ and to 0 if $x > n$. This is indeed the case because of our Lemma.
Now we suppose that $\sigma_c < \sigma + c \leq \sigma_a$. Let $\alpha > \sigma_a - \sigma$ and consider the rectangle γ_R with vertices $c + iT$, $\alpha + iT$, $\alpha - iT$, $c - iT$, for some T that will go to ∞.



By the Residue Theorem we have

$$\int_{\gamma_R} f(s+w) \frac{x^w}{w} dw = 0.$$

Since the integrant has no poles besides 0, which is not inside the rectangle.
Now, on the right side of the rectangle, f converges absolutely, and we already treated that

case. The left side is the integral we want to compute. To show that both integrals are equal, we only need to show that the integrals of the green horizontal sides of the rectangle tend to 0 as T goes to ∞. This can be done by using the vertical growth of dirichlet series (9):

$$f(s+w) = \mathcal{O}((t+\Im(w))^{1-(\sigma+c-\sigma_0)+\epsilon})$$

which is a $\mathcal{O}(\Im(w)^\gamma)$ whith $0<\gamma<1$. Hence $f(s+w)\frac{x^w}{w} = \mathcal{O}(|\Im(w)|^{\gamma-1})$, which shows that the horizontal integrals converges to 0 as T goes to ∞.

In conclusion, the integral of the red sides are equal, and we have proven the Perron Formula in the general case. □

2.2 Example on the Zeta function

Recall the Mangoldt function Λ that we introduced in the beginning, and its associated Dirichet series $F = -\frac{\zeta'}{\zeta}$. Because of the Perron formula, we know that

$$\sum_{n<x} \Lambda(n) = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'(w)}{\zeta(w)w}x^w dw.$$

We gave the idea of how to evaluate such an integral in the introduction. The obtained error term is not optimal, but it is a $o(x)$, which is sufficient for the prime number theorem. Now let us apply the same method to evaluate $\sum_{n<x} 1$.
We apply the Perron Formule to the Zeta function. We can take $c = 1+\epsilon$ and $s = 0$, which gives us:

$$\sum_{n<x} 1 = \frac{1}{2i\pi}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \zeta(w)\frac{x^w}{w}dw. \tag{15}$$

Now we cut the integral at $c+iT$ and $c-iT$, as in [Mur08], and we know the error term:

$$\int_{c-i\infty}^{c+i\infty} \frac{\zeta(w)}{w}x^w dw = \int_{c-iT}^{c+iT} \frac{\zeta(w)}{w}x^w dw + \mathcal{O}\Bigg(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c \min\bigg(1, T^{-1}\bigg|\log\frac{x}{n}\bigg|^{-1}\bigg)\Bigg).$$

The summation in the $\mathcal{O}$-term corresponding to $n < x/2$ or $n > 3x/2$ is bounded by $\mathcal{O}\bigg(\frac{x^c}{T}\bigg)$, and the summation corresponding to $x/2 \leqslant n \leqslant 3x/2$ is bounded by $\mathcal{O}\bigg(\frac{x}{T}\log(x)^2\bigg)$.
To evalute the remaining integral, we consider the curve $\mathcal{C}$ consisting of the three segments $[c+iT, 1/2+iT]$, $[1/2+iT, 1/2-iT]$ and $[1/2-iT, c-iT]$. By the residue Theorem we have

$$\int_{c-iT}^{c+iT} \frac{\zeta(w)}{w}x^w dw = x - \int_{\mathcal{C}} \frac{\zeta(w)}{w}x^w dw.$$

To estimate the integral over $\mathcal{C}$, we can use the estimate

$$\zeta(s) = \mathcal{O}(|t|^{1/2}), \text{ for } \sigma \geqslant 1/2 \text{ and when } |t| \to \infty.$$

We find that the integrals over the two honrizontal segments are $\mathcal{O}\bigg(\frac{x^c}{T^{1/2}}\bigg)$ and that the integral over the vertical segment is $\mathcal{O}\bigg(x^{1/2}T^{1/2}\log(T)\bigg)$.
After putting everything together we get

$$\sum_{n<x} 1 = x + \mathcal{O}\left(\frac{x^c}{T^{1/2}}\right) + \mathcal{O}\left(x^{1/2}T^{1/2}\log(T)\right).$$

We can take $T = x^{1/2}$ to make the error terms equal. In conclusion, we have the estimation

$$\sum_{n<x} 1 = x + \mathcal{O}\left(x^{3/4+\epsilon}\right).$$

We can observe that this method yields an error term which, although sufficient in the case of the prime number theorem, is not satisfactory. We shall use a different approach to get a better error term. The idea is to shift the integral of the Perron formula to the left to capture the poles of the integrand.

$$\int_{c-i\infty}^{c+i\infty} \frac{\zeta(w)}{w} x^w dw = \int_{\mathcal{C}'} \frac{\zeta(w)}{w} x^w dw + \int_{\delta-c-i\infty}^{\delta-c+i\infty} \frac{\zeta(w)}{w} x^w dw. \tag{16}$$

where $\mathcal{C}'$ is a curve containing all the poles of the integrand. The first integral can be computed explicitly by the Residue Theorem, and will give us the main term. For the second integral, we will use the functional equation of ζ to put it back on the right. Then it can be estimated and will give us an error term. In the next section, we are going to see the functional equation of ζ.

3 Functional equation of ζ

3.1 The Gamma function

We define the Gamma function in the half complex plane for every $\sigma > 0$:

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx. \tag{17}$$

The Gamma function has a central role in functional equations of Dirichlet series. Therefore, the study of its analytical properties is essential. Each proofs of the following results are detailed in [MV12] First of all, we shall extend Γ on the complex plane.

Theorem 2. *Γ can be extended to a meromorphic function on the whole complex plane. It has poles on every negative integers. If n is a negative integer, then its residue in $s = n$ is given by*

$$\underset{s=n}{Res}\, \Gamma(s) = \frac{(-1)^{-n}}{(-n)!}.$$

.

We will now give two asymptotic results of the Gamma function. The first one is the Stirling formula:

Theorem 3. *(Stirling) We have*

$$|\Gamma(x+iy)| \sim \exp\left(-\frac{1}{2}\pi|y|\right)|y|^{x-\frac{1}{2}}\sqrt{2\pi} \tag{18}$$

as y goes to ∞, uniformly in a vertical strip $x_1 \leqslant x \leqslant x_2$.

Theorem 4. *For any $\alpha \in \mathbb{C}$ and $\delta > 0$,*

$$\log\Gamma(z+\alpha) = (z+\alpha-1/2)\log z - z + \frac{1}{2}\log(2\pi) + \mathcal{O}\left(\frac{1}{|z|}\right) \tag{19}$$

as $|y| \to \infty$, the implicit constant being uniform in $-\pi+\delta \leqslant \arg(z) \leqslant \pi-\delta$.

3.2 The Zeta function and L functions

It is time to look at some examples of Dirichlet series. Consider the sequence $a_n = 1$ and the associated Dirichlet series. Because of the Riemann Criterion,, we now it has abcissa of convergence (and also absolute convergence) $\sigma_c = \sigma_a = 1$, defining a holomorphic function on the half plane $\sigma > 1$.

Definition 3.1. The function definied by

$$\zeta(s) = \sum_{n \geqslant 1} n^{-s} \tag{20}$$

for $\sigma > 1$ is called the *Riemann Zeta function.*

It is known that this function can be extended to a meromorphic function with an unique simple pole at $s = 1$, with residue 1. Moreover, $\zeta(s)$ is verifying the following functional equation:

Theorem 5. *(Functional equation of ζ) For all $s \in \mathbb{C}$ other than 0 and 1, we have*

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}. \tag{21}$$

Remark. This last equation can be written in a simpler way by considering a slight variation of the Zeta function $\zeta_2(s) = \sum_{n \geqslant 1} \frac{1}{n\sqrt{\pi}}$. Hence, we can generalize the definition of Dirichlet series by defining them as series of the form $\sum_{n \geqslant 1} a_n/\mu_n$ where $(\mu_n)_{n \in \mathbb{N}}$ is any increasing sequence going to ∞.

Remark. The functional equation (21) has two points worth mentioning. First of all, there is an axis of symmetry of abscissa 1/2. Secondly, there is the same Gamma function on both sides.

We can now take a look at Dirichlet L-functions, which have very similar functional equation.

Definition 3.2. For every $k \in \mathbb{N}^*$, we define a Dirichlet Character mod k as being a morphism χ from the multiplicative group $(\mathbb{Z}/k\mathbb{Z})^*$ into $\mathbb{C}^*$. Then, we can define a function $\hat{\chi} : \mathbb{Z} \to \mathbb{C}$ in the following way:

$$\hat{\chi(n)} = \begin{cases} \chi(n \bmod k) & \text{if } (n,k) = 1 \\ 0 & \text{else} \end{cases}.$$

Without confusion we shall denote $\hat{\chi}$ as just χ. The set of charaters modulo k forms a group. We denote χ_0 the trivial element of this group, called the principal character, when there is no ambiguity about k.

Remark. If χ is character modulo k, and if k' is a multiple of k, then we can define a character χ' modulo k' by

$$\chi'(n) = \begin{cases} \chi(n) & \text{if } (n,k') = 1 \\ 0 & \text{else} \end{cases}.$$

In this case we say that χ' is induced by χ. If a character is not induced by another character of inferior moduli we say it is primitive.

Definition 3.3. Let χ be a Dirichlet character, we define the associated L-function as the Dirichlet series

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

Since χ is completely multiplicative, we can develop $L(\chi, s)$ into an Euler product.

Proposition 9. *For every $\sigma > 1$,*

$$L(\chi, s) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}.$$

Since $|\chi(n)| \leq 1$ for all n and all characters χ, the abscissa of absolute convergence is always at most 1. The same can be said about the abscissa of convergence of $L(\chi_0, s)$, the L-function assiociated to the principal character. But we have more: using the Euler product, we can write

$$L(\chi_0, s) = \prod_{\substack{p \text{ prime} \\ p \nmid k}} (1 - p^{-s})^{-1} = \zeta(s) \prod_{p|k} (1 - p^{-s}).$$

Hence, we see that $L(\chi_0, s)$ can be extended to an analytical function over the complex plane, with an unique simple pole at $s = 1$. Furthermore, the residue of this pole can be computed and is equal to $\frac{\phi(k)}{k}$.

If $\chi \neq \chi_0$, then the oscillations of χ are sufficiently large to show that the sum $\sum_{n<x} \chi(n)$ is bounded for all x. This is enough to show that

Proposition 10. *If $\chi \neq \chi_0$ is a character modulo k, then $L(\chi, s)$ converges for all $\sigma > 0$ and defines an analytic function.*

Proof. Let $S(n) = \sum_{k \leqslant n} \chi(k)$ for all $n \in \mathbb{N}$. We apply an Abel transform for $N > M$:

$$\begin{aligned}
\left| \sum_{n=M}^{N} \frac{\chi(n)}{n^s} \right| &= \left| \sum_{n=M}^{N-1} S(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{S(N)}{N^s} - \frac{S(M-1)}{M^s} \right| \\
&\leqslant \mathcal{C} \left(\frac{1}{M^s} - \frac{1}{N^s} \right) + \frac{\mathcal{C}}{N^s} + \frac{\mathcal{C}}{M^s}.
\end{aligned}$$

If $\sigma > 0$, the left hand side can be arbitrarily small by choosing adequately M. Hence, by the Cauchy criterion, the series converges. □

Dirichlet L-functions can be extended to meromorphic functions on the whole complex plane. The proof is very similar to the one for the Zeta function.

Theorem 6. *Let χ be a primitive character. Then $L(\chi, s)$ can be extended to a meromorphic function. If $\chi(-1) = 1$, we have the following functional equation:*

$$\left(\frac{\pi}{k}\right)^{-\frac{1}{2}(1-s)} \Gamma\left(\frac{1-s}{2}\right) L(\bar{\chi}, 1-s) = \frac{k^{1/2}}{\tau(\chi)} \left(\frac{\pi}{k}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(\chi, s). \tag{22}$$

Where $\tau(\chi) = \sum_{m=1}^{k} \chi(m) e^{\frac{2i\pi m}{k}}$.

If $\chi(-1) = -1$, *we have*

$$\left(\frac{\pi}{k}\right)^{-\frac{2-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(\bar{\chi}, 1-s) = \frac{ik^{1/2}}{\tau(\chi)} \left(\frac{\pi}{k}\right)^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(\chi, s). \tag{23}$$

We shall give the main idea of the proof, without going too deep into details. A detailed proof is given in [Mun13].

Proof. Suppose $\chi(-1) = 1$. For every integer n we have

$$\begin{aligned}\left(\frac{k}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)n^{-s} &= \left(\frac{k}{\pi}\right)^{s/2}\left(\int_0^\infty e^{-x}x^{\frac{s}{2}-1}dx\right)n^{-s} \\ &= \int_0^\infty e^{-x}\left(\frac{kx}{\pi n^2}\right)^{s/2}\frac{dx}{x} \\ &= \int_0^\infty e^{-\frac{\pi n^2 x}{k}}x^{s/2}\frac{dx}{x}.\end{aligned}$$

To obtain the last line, we changed the variable by a dilatation of $\frac{\pi n^2}{k}$.
Now we can take the sum over n

$$\begin{aligned}\left(\frac{k}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(\chi,s) &= \int_0^\infty x^{s/2-1}\sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2 x}{k}}dx \\ &= \int_1^\infty x^{s/2-1}\sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2 x}{k}}dx + \int_0^1 x^{s/2-1}\sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2 x}{k}}dx \\ &= \int_1^\infty x^{s/2-1}\sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2 x}{k}}dx + \int_1^\infty x^{-s/2-1}\sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2}{kx}}dx.\end{aligned}$$

Put $\theta(x) = \sum_{n\geq 1}\chi(n)e^{-\frac{\pi n^2 x}{k}}$. By the Poisson formula, we can get a functional equation verified by θ, and thus deduce the functional equation for $L(\chi,s)$:

$$\tau(\bar{\chi})\theta(\chi,x) = \left(\frac{k}{x}\right)^{1/2}\theta(\bar{\chi},1/x)$$

for all $x > 0$.
Using this functional equation in the second integral above gives us

$$\left(\frac{k}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(\chi,s) = \int_1^\infty x^{s/2-1}\theta(\chi,x)dx + \frac{k^{1/2}}{\tau(\bar{\chi})}\int_1^\infty x^{-s/2-1/2}\theta(\bar{\chi},x)dx. \tag{24}$$

And by replacing s by $1-s$ and χ by $\bar{\chi}$,

$$\left(\frac{k}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(\bar{\chi},1-s) = \int_1^\infty x^{-s/2-1/2}\theta(\bar{\chi},x)dx + \frac{k^{1/2}}{\tau(\chi)}\int_1^\infty x^{s/2-1}\theta(\chi,x)dx. \tag{25}$$

Moreover, it can be shown that $|\tau(\chi)| = k^{1/2}$, so that equation (25) is just (24) multiplied by $\frac{k^{1/2}}{\tau(\chi)}$. This leads us to the functional equation (22).

Now if $\chi(-1) = -1$, we follow the same arguments but with some slight modifications. First of all we note that for all n,

$$\left(\frac{k}{\pi}\right)^{1+s/2}\Gamma\left(\frac{1+s}{2}\right)L(\chi,s) = \int_0^\infty\left(\sum_{n\geqslant 0}n\chi(n)e^{-\frac{-\pi n^2 x}{k}}\right)x^{(s-1)/2}dx.$$

Then put $\theta(x) = \sum_{n\geqslant 1}n\chi(n)e^{-\frac{\pi n^2 x}{k}}$. Again, we can use the Poisson formula to get a functional equation for θ:

$$\tau(\bar{\chi})\theta(\chi,x) = ik^{1/2}x^{-3/2}\theta(\bar{\chi},1/x).$$

And deduce from this the functional equation for $L(\chi,s)$. □

4 The main O-Theorem

Our goal is to study the sum of the first coefficients of a Dirichlet series. We will see during the proof of our main theorem that, in order to assure the convergence of some integrals, we need to add some weights to the coefficients. In section 4.1 we will talk about the Cesàro weights. Once we get the estimate for the Cesàro weights, we need to reduce the weights. This can be done by applying some operator, that will be the subject of section 4.2.

4.1 Cesàro weights

For every $k \in \mathbb{N}$, put

$$A_k(x) = \frac{1}{\Gamma(k+1)} \sum_{n \leqslant x} a_n (x-n)^k. \tag{26}$$

Note that $A_0(x)$ is just the sum $\sum_{n \leqslant x} a_n$. We have a result similar to the Perron Formula
propositionLet $x \in \mathbb{R}^+ \backslash \mathbb{N}$ and $c > 0$. We also suppose that $\sigma > \sigma_c - c$. Then

$$A_k(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) f(s)}{\Gamma(s+k+1)} x^{s+k} ds. \tag{27}$$

Proof. The proof is very similar to the proof of the Perron Formula. In the right hand side of (27), we espand f as its Dirichlet series and invert the integral and the sum.

$$\sum_{n=1}^{\infty} a_n \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(s+k+1) n^s} x^{s+k} ds.$$

Take the nth term of the sum, and put $\alpha = x/n$

$$a_n x^k \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(s+k+1)} \alpha^s ds. \tag{28}$$

If $\alpha > 1$, then the integral is equal to 0 by considering the left half circle centered in c and with radius R, as in the proof of Perron's formula. There are no poles in this circle, so the integral is equal to 0.

If $\alpha < 1$, we consider C_R the right half circle centered in c and with radius R. If R is sufficently large, this circle contains all the poles of the integrant, which are all integers in $]-k-1, 0]$.
It can be shown that the integral is equal to the sum on the residues. Since all the poles are of order 1, the residues can easily be computed. Let l be an integer in $[0, k+1[$, so that $-l$ is a pole of the integrant, then

$$\begin{aligned}
\operatorname*{Res}_{s=-l}\left(\frac{\Gamma(s)}{\Gamma(s+k+1)} \alpha^s\right) &= \lim_{s \to -l} (s+l) \frac{\Gamma(s)}{\Gamma(s+k+1)} \alpha^s \\
&= \frac{\alpha^{-l}}{\Gamma(-l+k+1)} \lim_{s \to -l} (s-l)\Gamma(s) \\
&= \frac{\alpha^{-l}}{\Gamma(-l+k+1)} \frac{(-1)^l}{l!} \\
&= \frac{1}{\Gamma(k+1)\alpha^k} \frac{\Gamma(k+1)}{\Gamma(k+1-l)l!} \alpha^{k-l} (-1)^l
\end{aligned}$$

Hence (28) is equal to the sum

$$a_n x^k \frac{1}{\Gamma(k+1)\alpha^k} \sum_{l=0}^{\lceil k+1 \rceil - 1} \frac{\Gamma(k+1)}{\Gamma(k+1-l)l!} \alpha^{k-l}(-1)^l$$

We recognize a generalized binomial formula, giving us

$$a_n x^k \frac{1}{\Gamma(k+1)\alpha^k}(\alpha - 1)^k$$

After some simplifications, we finally get

$$\frac{a_n}{\Gamma(k+1)}(x-n)^k$$

which correspond to the general term of the sum in (26). □

4.2 The operator Δ_y^ρ

In this section we shall introduce a very useful operator for our theorem. We start by giving a very analytic definition.

Definition 4.1. Let ρ be an integer and F a function of class C^ρ. For every $y > 0$, we define the finite difference by

$$\Delta_y^\rho F(x) = \int_x^{x+y} \int_{t_1}^{t_1+y} \int_{t_2}^{t_2+y} \dots \int_{t_{\rho-1}}^{t_{\rho-1}+y} F^{(\rho)}(t_\rho) dt_\rho dt_{\rho-1} \cdots dt_1. \tag{29}$$

Remark. Δ_y^0 is the trivial operator and $\Delta_y^1 F(x) = F(y) - F(x)$.
Hence we see that Δ_y^1 acts like a discrete derivation.

Of course, because of the linear nature of the integral and the derivative, it is not difficult to show that

Proposition 11. *The operator Δ_y^ρ is linear.*

It is interesting to note that this operator has also a more arithmetical definition.

Proposition 12. *We have*

$$\Delta_y^\rho F(x) = \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} F(x + \nu y). \tag{30}$$

Proof. We will proceed by induction. The case $\rho = 0$ is trivial. Let us suppose that (30) is true for some ρ.

$$\begin{aligned}
\Delta_y^{\rho+1} F(x) &= \int_x^{x+y} \int_{t_1}^{t_1+y} \int_{t_2}^{t_2+y} \dots \int_{t_{\rho-1}}^{t_\rho+y} F^{(\rho+1)}(x) dt_{\rho+1} dt_\rho \dots dt_1 \\
&= \int_x^{x+y} \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} F'(t_1 + \nu y) dt_1 \\
&= \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} \Big(F(x + (\nu+1)y) - F(x + \nu y) \Big) \\
&= \sum_{\nu=1}^{\rho+1} (-1)^{\rho-\nu-1} \binom{\rho}{\nu-1} F(x+\nu y) + \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu-1} \binom{\rho}{\nu} F(x+\nu y) \\
&= \sum_{\nu=0}^{\rho+1} (-1)^{\rho-\nu+1} \binom{\rho+1}{\nu} F(x+\nu y).
\end{aligned}$$

Hence the proposition is true for all integer ρ. □

Remark. This last definition is more general than the first one, since it covers all functions, and not just functions of class C^ρ. However, we will only be dealing with holomorphic functions, therfore both definitions can be considered as equivalent.

The next proposition shows how the Δ_y^ρ operator enable us to reduce the weigths.

Proposition 13. *For every integer ρ and $y > 0$*

$$\Delta_y^\rho A_\rho(x) = A_0(x)y^\rho + \mathcal{O}\bigg(y^\rho \sum_{x<\lambda_n<x+\rho y} |a_n|\bigg). \tag{31}$$

Proof. We start by splitting the sum and applying the linear proprety of the Δ operator.

$$\begin{aligned}
\Delta_y^\rho A_\rho(x) &= \Delta_y^\rho\bigg(\sum_{\lambda_n<x} a_n \frac{(x-\lambda_n)^\rho}{\Gamma(\rho+1)}\bigg) \\
&= \sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu} \sum_{\lambda_n<x+\nu y} a_n \frac{(x+\nu y-\lambda_n)^\rho}{\Gamma(\rho+1)} \\
&= \sum_{\lambda_n<x}\sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu} a_n \frac{(x+\nu y-\lambda_n)^\rho}{\Gamma(\rho+1)} + \sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu} \sum_{x<\lambda_n<x+\nu y} a_n \frac{(x+\nu y-\lambda_n)^\rho}{\Gamma(\rho+1)} \\
&= \sum_{\lambda_n<x} a_n \frac{\Delta_y^\rho(x-\lambda_n)^\rho}{\Gamma(\rho+1)} + \sum_{\nu=0}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu} \sum_{x<\lambda_n<x+\nu y} a_n \frac{(x+\nu y-\lambda_n)^\rho}{\Gamma(\rho+1)}.
\end{aligned}$$

Then, since the ρth derivative of $(x-\lambda_n)^\rho$ is $\Gamma(\rho+1)$, it is easy to see $\frac{\Delta_y^\rho(x-\lambda_n)^\rho}{\Gamma(\rho+1)} = y^\rho$.
Furthermore, the dominant term in the second sum is when $\nu = \rho$. This term is a $\mathcal{O}\bigg(\sum_{x<\lambda_n<x+\rho y})|a_n|y^\rho\bigg)$.
This is exactly the $\mathcal{O}$ term in (31). □

This proposition is very interesting, because it shows that in order estimate the sum of the first coefficients of a Dirichlet series, we can estimate any Cesaro weight A_ρ and then apply the operator Δ_y^ρ. This will be the general idea of the proof of our main $\mathcal{O}$ Theorem.
Finally, we shall give the behavior of $\Delta_y^\rho F(x)$ for some function F.

Proposition 14. *Let F be a function of class C^∞, and suppose that $F(x) = O(x^\alpha)$ and that $F^{(\rho)}(x) = O(x^\beta)$. We also suppose that $y = O(x)$. Then*

$$\Delta_y^\rho F(x) = \left\{ \begin{array}{c} \mathcal{O}(x^\alpha) \\ \mathcal{O}(y^\rho x^\beta) \end{array} \right. . \tag{32}$$

Proof. To get the first term we use equation (30). Since the sum is finite and
$F(x+\nu y) = O((x+\nu y)^\alpha) = O(x^\alpha)$, we have $\Delta_y^\rho F(x) = O(x^\alpha)$.
On the other hand, we can give an estimate for Δ_y^ρ by using equation (29). Indeed, the integrand is dominated by $(x+\rho y)^\beta$, which has same order as x^β since $y = O(x)$. Since we take the integrals over segments of lengths y, the result is dominated by $y^\rho x^\beta$. □

4.3 Capturing the poles

Let ϕ be a Dirichlet series, we shall denote

$$Q_0(x) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{\phi(s)}{s} x^s ds,$$

where $\mathcal{C}$ is a curve which encloses all the poles of the integrand. Note that $Q_0(x)$ correspond to the first term on the right hand side of (16). It is the integral that we get when we shift the line of integration in the Perron formula.
Since we will be dealing with more general Cesaro weights, we shall consider the following definition:

Definition 4.2. For every integer ρ, let

$$Q_\rho(x) = \frac{1}{2i\pi}\int_{C_\rho}\frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)}x^{s+\rho}ds, \tag{33}$$

where $\mathcal{C}_\rho$ is a curve containing all the poles of the integrant.

Because of the Residue Theorem, we can compute the value of $Q_0(x)$ with the residues of the integrand. This is done in the folowing proposition.

Proposition 15. *For every pole ξ of $\frac{\phi(s)}{s}x^s$, we have*

$$\underset{s=\xi}{Res}\left(\frac{\phi(s)}{s}x^s\right) \asymp x^{\Re(\xi)}log(x)^{r_\xi - 1} \tag{34}$$

where r_ξ is the order of the pole.

Proof. Let's develop ϕ, x^s and $\frac{1}{s}$ into their Laurent expansion:

- $\phi(s) = \sum_{n\geqslant -r_\xi} a_n(s-\xi)^n$
- $x^s = x^\xi \sum_{n\geqslant 0}\frac{1}{n!}(s-\xi)^n log(x)^n$
- $\frac{1}{s} = \sum_{n\geqslant 0} c_n(s-\xi)^n$

Using Theorem 5, we can compute the residue of the product at ξ:

$$\underset{s=\xi}{\text{Res}}\left(\frac{\phi(s)}{s}x^s\right) = x^\xi \sum_{k+l+m=-1}\frac{a_k}{l!}log(x)^l c_m. \tag{35}$$

The biggest term in this sum is when $k=-r_\xi$, $l=r_\xi-1$ and $m=0$. Since coefficients a_n and c_n only depends of ϕ, we conclude that $\text{Res}_\xi(\frac{\phi(s)}{s}x^s) \asymp x^{\Re(\xi)}log(x)^{r_\xi-1}$. □

In conclusion we have a good estimate for Q_0:

$$Q_0(x) \asymp \sum_\xi x^{\Re(\xi)}log(x)^{r_\xi-1}. \tag{36}$$

Since the ρ^{th} derivative of Q_ρ is Q_0, we have by definition

$$\Delta_y^\rho Q_\rho(x) = \int_x^{x+y}\int_{t_1}^{t_1+y}...\int_{t_{\rho-1}}^{t_{\rho-1}+y} Q_0(t_\rho)dt_\rho...dt_1 \tag{37}$$

This gives us a relation between Q_ρ and Q_0.

Proposition 16. *Assume $y = O(x)$, then*

$$\Delta_y^\rho Q_\rho(x) = Q_0(x)y^\rho + \mathcal{O}\left(x^{q-1}(log^{r-1}x)y^{\rho+1}\right). \tag{38}$$

where q is the largest real part of a pole with greatest real part, and r is the maximum order of a pole of real part q.

Proof. We start by writing $Q_0(t_\rho) = Q_0(x) + \int_x^{t_\rho} Q_0'(s)ds$.
Note that $Q_0'(s) \asymp \sum_\xi \left(q s^{q-1} log^{r-1} s + s^{q-1}(r-1) log^{r-2} s \right) \asymp s^{q-1} log^{r-1} s$.
By putting this into (37)

$$\Delta_y^\rho Q_\rho(x) = Q_0(x) y^\rho + \mathcal{O}\left(\int_x^{x+y} \int_{t_1}^{t_1+y} ... \int_{t_{\rho-1}}^{t_{\rho-1}+y} \int_x^{t_\rho} s^{q-1} \log^{r-1}(s) ds dt_\rho ... dt_1 \right).$$

The integrant in the $\mathcal{O}$-term is smaller than $(x+y\rho)^{q-1} \log^{r-1}(x+y\rho)$. Since $y = \mathcal{O}(x)$, this has the same order of magnitude as $x^{q-1}\log^{r-1}(x)$. After taking $\rho+1$ integrals over segments of lenghts smaller than $y\rho$, we finally get the $\mathcal{O}$-term in equation (16). □

4.4 The main theorem

We will now consider two general Dirichlet series $\phi(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s}$ and $\psi(s) = \sum_{n \geq 1} \frac{b_n}{\mu_n^s}$, where (λ_n) and (μ_n) are two increasing sequences of real numbers going to ∞. We also assume that they satisfy the following functional equation:

$$\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s) \tag{39}$$

where $\Delta(s) = \prod_{\nu=1}^N \Gamma(\alpha_\nu s + \beta_\nu)$ is a product of gamma functions with $\alpha_\nu \in \mathbb{R}$ and $\beta_\nu \in \mathbb{C}$. δ is a positive real number.
Before giving the main Theorem, we shall introduce the following notations:

- $A = \sum_\nu \alpha_\nu$.
- $\beta \in \mathbb{R}$ is such that $\psi(\beta)$ converges absolutely.
- q is the real part of the pole of ϕ with gratest real part.
- r is the maximum order of the poles with real part q.
- $t \leqslant 1/2$ is such that $\frac{t}{2A} < 1$ and there exist an integer n such that $0 < \{\frac{n+A\delta}{2A}\} \leqslant 1 - \frac{t}{2A}$.
- $u = \beta - \delta/2 - t/2A$.

The following Theorem was proven by Chandrasekharan and Narasimhan in their article [CN62].

Theorem 7. *Suppose that all the singularities of ϕ, as defined above, are poles and that there is a finite number of them. Assume also that $A > 0$, then we have*

$$\sum_{n<x} a_n - Q_0(x) = \mathcal{O}\left(x^{\delta/2 - t/2A + 2A\eta u} \right) + \mathcal{O}\left(x^{q-1/2A-\eta} log^{r-1} x \right) + \mathcal{O}\left(\sum_{x<n<x'} |a_n| \right)$$

where $\eta \geqslant 0$ and $x' = x + \mathcal{O}\left(x^{1-\eta-1/2A} \right)$.

Chandrasekharan and Narasimhan treated the case $A \geqslant 1$, where t can awlays be chosen to be $1/2$. We propose this generalization for the case where $A > 0$. We shall prove this Theorem in three steps. First of all, we will write $A_\rho(x) - Q_\rho(x)$ as a sum $\sum_{n \geqslant 1} \frac{b_n}{\nu_n^{\delta+\rho}} I(xv_n)$, where I is a function defined by an integral. Secondly, we will give an estimate for this function I and for $I^{(\rho)}$. This estimate will give us the first $\mathcal{O}$-term. Finally, we shall use our operator Δ_y^ρ to deduce a formula for A_0 and Q_0. This operation will cost us the two ther $\mathcal{O}$-terms.

4.4.1 Evaluating $A_\rho - Q_\rho$

For $\rho \geqslant 0$, Let

$$A_\rho(x) = \frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \leqslant x} a_n (x-\lambda_n)^\rho.$$

For c sufficiently large, that we may fix later, we have

$$A_\rho(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho} ds.$$

If c is large enough, we may also assume that all the poles of the integrant are in the strip $\delta - c < \sigma < c$. Hence, because of the Residue Theorem, we have

$$\frac{1}{2i\pi} \int_{T_R} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho} ds = Q_\rho(x) \tag{40}$$

where T_R is the rectangle with vertices at $c \pm iR$ and $\delta - c \pm iR$, and

$$Q_\rho = \frac{1}{2i\pi} \int_{C_\rho} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho} ds$$

where C_ρ encloses the same singularities as the strip $\delta - c < \sigma < c$.
In equation (40), we can let R go to ∞ to get

$$A_\rho(x) - Q_\rho(x) = \frac{1}{2i\pi} \int_{\delta-c-i\infty}^{\delta-c+i\infty} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho} ds.$$

Now we can change the variable with $s' = \delta - s$ and then apply functional equation (39):

$$A_\rho(x) - Q_\rho(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\delta-s)\Delta(s)\psi(s)}{\Gamma(\delta-s+\rho+1)\Delta(\delta-s)} x^{\delta-s+\rho} ds. \tag{41}$$

Now put

$$I(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\delta-s)\Delta(s)}{\Gamma(\delta-s+\rho+1)\Delta(\delta-s)} x^{\delta-s+\rho} ds, \tag{42}$$

so that

$$A_\rho(x) - Q_\rho(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\delta-s)\Delta(s)}{\Gamma(\delta-s+\rho+1)\Delta(\delta-s)} x^{\delta-s+\rho} \sum_{n\geqslant 1} \frac{b_n}{\mu_n^s} \tag{43}$$

$$= \frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{b_n}{\mu_n^{\delta+\rho}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\delta-s)\Delta(s)}{\Gamma(\delta-s+\rho+1)\Delta(\delta-s)} (x\mu_n)^{\delta-s+\rho} \tag{44}$$

$$= \frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{b_n}{\mu_n^{\delta+\rho}} I(x\mu_n). \tag{45}$$

To assure the convergence of the integral in (41), we need to study the asymptotic behavior of the integrand. This can be done by using the Stirling formula (18).
The integral converges if and only if

$$2Ac - A\delta < \rho. \tag{46}$$

Thus, by chosing ρ sufficiently large, the previous calculations are justified.
Moreover, to justify (45), we also need to take ρ sufficiently large. In the next we will study the asymptotic behavior of I to find the lower bound of ρ and also to estimate $A_\rho - Q_\rho$.

4.4.2 Estimate for $I(x)$ and $I^{(\rho)}(x)$

We shall now study the asymptotic behavior of the integrand in (42). In order to do that, we will first take the log of the integrant and apply (19). We get

$$\log G(s) - \log\left(\frac{\Gamma(As+\mu)}{\Gamma(\lambda - As)} e^{\Theta s}\right) = B + \mathcal{O}\left(\frac{1}{|s|}\right). \tag{47}$$

Where

$$\begin{aligned} G(s) &= \frac{\Gamma(\delta - s)\Delta(s)}{\Gamma(\rho + 1 + \delta - s)\Delta(\delta - s)}, \\ \mu &= 1/2 + \sum_{\nu=1}^{N} (\beta_\mu - 1/2), \\ \lambda &= \mu + A\delta + \rho + 1, \\ \Theta &= 2(\sum_{\nu=1}^{N} \alpha_\nu \log(\alpha_\nu) - A\log A), \\ B &= -\delta \sum_{\nu=1}^{N} \alpha_\nu \log \alpha_\nu + (A\delta + \rho + 1)\log A. \end{aligned}$$

Now take the exponential on boh sides:

$$G(s) = H(s) e^{\mathcal{O}(\frac{1}{|s|})}$$

where $H(s) = \frac{\Gamma(As+\mu)}{\Gamma(\lambda - As)} e^{B+\Theta s}$.

Finally, by subtracting $H(s)$ on both sides and using $e^{\mathcal{O}(\frac{1}{|s|})} - 1 = \mathcal{O}\left(\frac{1}{|s|}\right)$, we get

$$G(s) - H(s) = H(s)\mathcal{O}\left(\frac{1}{|s|}\right).$$

Now we can write $I(x)$ by using the new function H.

$$\begin{aligned} I(x) &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} G(s) x^{\delta+\rho-s} ds \\ &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left\{ H(s)\mathcal{O}\left(\frac{1}{|s|}\right)\right\} x^{\delta+\rho-s} ds + \int_{c-i\infty}^{c+i\infty} H(s) x^{\delta+\rho-s} ds. \end{aligned}$$

The second integral can be computed explicitely by using Bessel functions:

$$\frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} H(s) x^{\delta+\rho-s} ds = \mathcal{C}(y^{1/(2A)})^{A\delta + (2A-1)\rho} \mathcal{J}_{2\mu + A\delta + \rho}(2y^{1/(2A)})$$

where $\mathcal{J}$ is the Bessel function.
Thus, by using the asymptotic behavior of $\mathcal{J}$, which is of order $x^{-1/2}$, we conclude that

$$\left| \int_{c-i\infty}^{c+i\infty} H(s) x^{\delta+\rho-s} ds \right| = \mathcal{O}\left(x^{\delta/2 + (1-1/(2A))\rho - \frac{1}{4A}} \right).$$

On the other hand, we can shift the line of integration of the first integral until we hit a pole. The poles of the integrant are the $\delta + k$ where k is a positive integer. Hence, by shifting the

integral to the right, the first pole we will hit is $\delta+\nu_\rho$ where ν_ρ is the smallest integer bigger than $\frac{\rho-A\delta}{2A}$. By hypothesis, we can choose ρ so that $\nu_\rho \geq \frac{\rho-A\delta+t}{2A}$.
Thus we have

$$\int_{c-i\infty}^{c+i\infty}\left\{H(s)\mathcal{O}\left(\frac{1}{|s|}\right)\right\}x^{\delta+\rho-s}ds = \int_{c+t/(2A)-i\infty}^{c+t/(2A)+i\infty}\left\{H(s)\mathcal{O}\left(\frac{1}{|s|}\right)\right\}x^{\delta+\rho-s}ds$$
$$= \mathcal{O}\left(x^{(A\delta+(2A-1)\rho-t)/2A}\right).$$

In conclusion:

$$I(x) = \mathcal{O}\left(x^{(A\delta+(2A-1)\rho-t)/2A}\right). \tag{48}$$

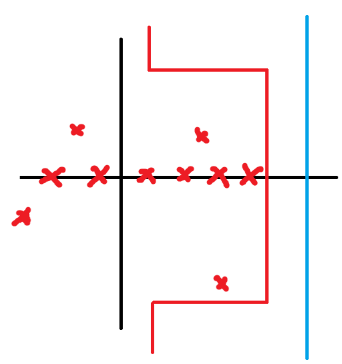
At this point we are able to establish that for (45) to hold, we need to have

$$\rho \geqslant 2A\beta - A\delta - t.$$

ρ can be chosen arbitrarely large, so this is not a problem. Furthermore, this also shows that we can't have $\delta/2 + t/(2A) \geq \beta$, because then (45) would be true for $\rho = 0$, which is clearly not the case since $A_0(x)$ is not continuous. Hence, we have

$$\beta - \delta/2 - t/(2A) > 0. \tag{49}$$

To get an estimate for $I^{(\rho)}(x)$, we need to derive (42). However, this is not always possible because the abscissa of convergence c has to verify (46). Hence, we shall first shift the line of integration, in order to later derive inside the integral. We shift the vertical line to a curve consisting of the lines $c_0 + it$ with $|t| > R$, and the three sides of the rectangle with vertices c_0-iR,c_0+iR,C_0+r+iR and c_0+r-iR, for some R and some r and with $c_0 = \frac{\delta}{2}$. During the shift, we have to be careful to not cross any poles. This can be done by chosing adequately R and r.



In the image, we shift the blue vertical line to the red one, by taking care to not pass by any red cross, which are the singularities of the integrant.
After derivating inside the integral we have

$$I^{(\rho)}(x) = \frac{1}{2i\pi}\int_{\mathcal{C}} G(s)x^{\delta-s}ds.$$

We can now use the same method we used to estimate $I(x)$ to get the estimation:

$$I^{(\rho)}(x) = \mathcal{O}\left(x^{\delta/2-\frac{t}{2A}}\right). \tag{50}$$

4.4.3 Reducing the weights

The last step consists at applying the operator Δ_y^ρ on both sides of (45) to get a result for A_0 and Q_0. By the linearity of the operator, we get

$$\Delta_y^\rho A_\rho(x) - \Delta_y^\rho Q_\rho(x) = \frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{b_n}{\nu_n^{\delta+\rho}} \Delta_y^\rho I(x\nu_n). \tag{51}$$

Because of (38) and (31), we have

$$y^\rho\left[A_0(x) - Q_0(x)\right] + \mathcal{O}\left(x^{q-1} log^{r-1} x y^{\rho+1}\right) + \mathcal{O}\left(y^\rho \sum_{x<\lambda_n<x+\rho y} |a_n|\right) = \frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{b_n}{\mu_n^{\delta+\rho}} \Delta_y^\rho I(x\mu_n).$$

Now we shall give the behavior of the right hand side. We can use (32) to give an estimate of $\Delta_y^\rho I(x\mu_n)$.

$$\Delta_y^\rho I(x\mu_n) = \begin{cases} O((x\mu_n)^{\delta/2-t/(2A)+\rho(1-1/(2A))}) \\ O(y^\rho x^{\delta/2-t/(2A)} \mu_n^{\delta/2-t/(2A)+\rho}) \end{cases}.$$

For some $z \in \mathbb{R}$ that we might fix later, we can split the sum on the right hand side:

$$\begin{aligned} \frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{b_n}{\mu_n^{\delta+\rho}} \Delta_y^\rho I(x\mu_n) &= \frac{1}{2i\pi} \sum_{\mu_n\leqslant z} \frac{b_n}{\mu_n^{\delta+\rho}} \Delta_y^\rho I(x\mu_n) + \sum_{\mu_n>z} \frac{b_n}{\mu_n^{\delta+\rho}} \Delta_y^\rho I(x\mu_n) \\ &= \mathcal{O}\left(\sum_{\mu_n\leqslant z} \frac{|b_n|}{\mu_n^{\delta/2+t/(2A)}} y^\rho x^{\delta/2-t/(2A)}\right) \\ &\quad + \mathcal{O}\left(\sum_{\mu_n>z} \frac{|b_n|}{\mu_n^{\delta/2+\rho/(2A)+t/(2A)}} x^{\delta/2-t/(2A)+\rho(1-1/(2A))}\right). \end{aligned}$$

Moreover, we know that $\delta/2+t/(2A) < \beta$ by (49), and since ρ can be chosen to be arbitrarily large, we may also assume that $\delta/2 + t/(2A) + \rho/(2A) > \beta$. Hence, on the one hand we have

$$\begin{aligned} \sum_{\mu_n>z} \frac{|b_n|}{\mu_n^{\delta/2+t/(2A)+\rho/(2A)}} &= \sum_{\mu_n>z} \frac{|b_n|}{\mu_n^{\delta/2+t/(2A)+\rho/(2A)-\beta} \mu_n^\beta} \\ &\leqslant z^{\beta-\delta/2-t/(2A)-\rho/(2A)} \sum_{\mu_n>z} \frac{|b_n|}{\mu_n^\beta} = \mathcal{O}\left(z^{\beta-\delta/2-t/(2A)-\rho/(2A)}\right), \end{aligned}$$

while on the other hand

$$\sum_{\mu_n\leqslant z} \frac{|b_n|}{\mu_n^{\delta/2+t/(2A)}} = \sum_{\mu_n\leqslant z} \frac{|b_n|\mu_n^{\beta-\delta/2-t/(2A)}}{\mu_n^\beta} \leqslant z^{\beta-\delta/2-t/(2A)} \sum_{\mu_n\leqslant z} \frac{|b_n|}{\mu_n^\beta} = \mathcal{O}\left(z^{\beta-\delta/2-t/(2A)}\right).$$

Thus we conclude that:

$$\frac{1}{2i\pi} \sum_{n\geqslant 1} \frac{|b_n|}{\mu_n^{\delta+\rho}} \Delta_y^\rho I(x\mu_n) = \mathcal{O}\left(y^\rho x^{\delta/2-t/(2A)} z^u\right) + \mathcal{O}\left(x^{\delta/2-t/(2A)+\rho(1-1/(2A))} z^{u-\rho/(2A)}\right)$$

where $u = \beta - \delta/2 - 1/(4A)$. Now we have to make a compromise between the two $\mathcal{O}$-terms. We still have a choice to make for y and z. We can start by setting

$$z = \frac{x^{2A-1}}{y^{2A}}.$$

Which gives us

$$\frac{1}{2i\pi}\sum_{n\geqslant 1}\frac{b_n}{\mu_n^{\delta+\rho}}\Delta_y^\rho I(x\mu_n) = \mathcal{O}\left(y^{\rho-2Au}x^{\delta/2-t/(2A)+(2A-1)u}\right).$$

Putting this into (51) and using (38) and (31) we get

$$\begin{aligned}A_0(x)-Q_(x) =&\mathcal{O}(y^{-2Au}x^{\delta/2-t/2A+(2A-1)u})\\ &+\mathcal{O}(yx^{q-1}\log^{r-1}x)\\ &+\mathcal{O}(\sum_{x<\lambda_n\leqslant x+\rho y}|a_n|).\end{aligned}$$

And put

$$y = x^{1-1/(2A)-\eta}$$

in order to simplify the first $\mathcal{O}$-term. We keep the choice of $\eta>0$ open. In some cases, η could be chosen to optimize the three $\mathcal{O}$-terms. □

4.5 Dirichlet series with positive coefficients

In a lot of cases the coefficients of a Dirichlet series are positive. This is the case for ζ or for $\sum\frac{d(n)}{n^s}$. The function $A_0(x)$ is therefore monotone, and thus we can get a better estimate.

Theorem 8. *Under the same conditions as in Theorem 7, and with the additional condition that $a_n\geqslant 0$ for all $n\geqslant 1$, we have*

$$A_0(x)-Q_0(x) = \mathcal{O}\left(x^{\delta/2-t/(2A)+2A\eta u}\right)+\mathcal{O}\left(x^{q-1/(2A)-\eta}\log(x)^{r-1}\right). \tag{52}$$

Recall that $Q_0(x)\asymp x^q\log(x)^{r-1}$. Hence, we can observe in the second $\mathcal{O}$-term that we gain $1/(2A)$ compared to the main term.

Proof. Since $A_0(x)$ is monotone, we have for all $t\in[x,x+\rho y]$

$$A_0(x)\leqslant A_0(t)\leqslant A_0(x+\rho y)$$

so that, after taking the integral

$$y^\rho A_0(x)\leqslant \Delta_y^\rho A_\rho(x)\leqslant y^\rho A_0(x+\rho y)$$

Now divide by y^ρ and substract $Q_0(x)$:

$$\begin{aligned}A_0(x)-Q_0(x) &\leqslant y^{-\rho}\Delta_y^\rho A_\rho(x) - y^{-\rho}\Delta_y^\rho Q_\rho(x)+\mathcal{O}\left(x^{q-1}\log(x)^{r-1}y\right)\\ &= \mathcal{O}\left(x^{\delta/2-1/(4A)+2A\eta u}\right)+\mathcal{O}\left(x^{q-1/(2A)-\eta}\log(x)^{r-1}\right)\\ &\leqslant A_0(x+\rho y)-Q_0(x).\end{aligned}$$

Furthermore, we can see that

$$Q_0(x+\rho y)-Q_0(x) = \mathcal{O}\left(x^{q-1}y\log(x)^{r-1}\right),$$

which is just a $\mathcal{O}\left(x^{q-1/(2A)-\eta}\log(x)^{r-1}\right)$ by taking $y = \mathcal{O}\left(x^{1-1/(2A)-\eta}\right)$.
Thus we have the proper bounding to conclude

$$A_0(x) - Q_0(x) = \mathcal{O}\left(x^{\delta/2-t/(2A)+2A\eta u}\right) + \mathcal{O}\left(x^{q-1/(2A)-\eta}\log(x)^{r-1}\right).\square$$

If two Dirichlet series ϕ and ψ satisfy all the conditions of Theorem 8, and if the greatest order of a pole with greatest real part of ϕ is 1, then we can take

$$\eta = \frac{t + 2Aq - A\delta - 1}{4A^2u + 2A}$$

to make the two $\mathcal{O}$-terms equal. In this cas, we have

$$A_0(x) - Q_0(x) = \mathcal{O}\left(x^{q-1/(2A)-\frac{t+2Aq-A\delta-1}{4A^2u+2A}}\right).$$

In their article, ChandraseKharan and Narasimhan presented the case where $A \geq 1$, and thus $t = 1/2$. We can now take a look at some examples where $A = 1/2$, as for ζ or L-functions.

4.6 Some examples

The Zeta function satisfies (39) and has positive coefficients. In order to apply Theorem 8, we should take $\phi(s) = \psi(s) = \pi^{-s/2}\zeta(s)$.
The parameters are $\delta = 1$, $A = 1/2$, $q = 1$ and $r = 1$. For every integer n, $\{\frac{n-A\delta}{2A}\} = 1/2$, hence we can take $t = 1/2$. For β we can take $1+\epsilon$, so that $u = \epsilon$, and

$$A_0(x) - Q_0(x) = \mathcal{O}\left(x^{\eta}\epsilon\right) + \mathcal{O}\left(x^{-\eta}\right).$$

Hence, with $\eta = 0$

$$A_0(x) - Q_0(x) = \mathcal{O}\left(1\right).$$

Of course, $A_0(x)$ is just a sum of 1's, and is therefore equal to $[x\pi^{-1/2}]$. On the other hand, $\pi^{-s/2}\zeta(s)$ has only a simple pole at $s = 1$, hence $Q_0(x) = \pi^{-1/2}x$. So we find what we would expect.

Let χ be a primitive, non trivial character modulo k. We have seen that the L-functions $L(\chi, s)$ and $L(\bar{\chi}, s)$ are related by equation (22). We can apply Theorem 7 with $\phi(s) = \frac{k^{1/2}}{\tau(\chi)}(\frac{\pi}{k})^{-s/2}L(\chi, s)$ and $\psi(s) = (\frac{\pi}{k})^{-s/2}L(\bar{\chi}, s)$. The parameters are the same as for the Zêta function, and we also get

$$A_0(x) - Q_0(x) = \mathcal{O}\left(1\right).$$

Here we have

$$A_0(x) = \frac{k^{1/2}}{\tau(\chi)} \sum_{n(\frac{\pi}{k})^{1/2} < x} \chi(n),$$

and since $L(\chi, s)$ has no poles, $Q_0(x)$ is a constant. Hence we conclude that

$$\sum_{n<x} \chi(n) = \mathcal{O}\left(1\right).$$

Which corresponds to what we should find.
We now give an example of a less trivial problem. We want to give an estimation of the sum $\sum_{n<x} d(n)$. We can start by giving a simple estimation by writing

$$\sum_{n<x} d(n) = \sum_{n<x} \sum_{ab=n} 1 = \sum_{a<x} [x/n] = \sum_{a<x} \Big(x/n + \mathcal{O}(1) \Big) = x\log(x) + \mathcal{O}(x).$$

Now we apply Theorem 8 to see what error term we obtain. The main term is given by

$$Q_0(x) = \int_{\mathcal{C}} \frac{\zeta(s)^2}{s} x^s ds,$$

where $\mathcal{C}$ is a curve containing the poles of the integrand, which are 1 and 0. The residue at 0 is a constant. For the residue at 1, we can write the following Laurent expansions:

- $\zeta(s)^2 = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \dots$
- $x^s = x + x\log(x)(s-1) + \dots$
- $\frac{1}{s} = 1 - (s-1) + \dots$

for $|s-1| < 1$, and where γ is the Euler constant.
Hence the residue of the integrand at $s = 1$ is $x\log(x) + (2\gamma - 1)x$. Moreover, ζ^2 satisfies

$$\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s) = \pi^{-(1-s)}\Gamma^2\left(\frac{1-s}{2}\right)\zeta^2(1-s).$$

We are now able to apply Theorem 8 with $A = 1$, $q = 1$, $r = 2$, $\delta = 1$, $\beta = 1 + \epsilon$ and we might choose $\eta = 1/6$. We conclude that

$$\sum_{n<x} d(n) = x\log(x) + x(2\gamma - 1) + \mathcal{O}\big(x^{1/3+\epsilon}\big).$$

The smallest value of θ such that $\sum_{n<x} d(n) = x\log(x) + x(2\gamma - 1) + \mathcal{O}\big(x^{\theta+\epsilon}\big)$ for all $\epsilon > 0$ is still unknown, but it is conjectured to be 1/4. This problem is the Dirichlet divisor problem. Here we have shown that $\theta \leqslant 1/3$.

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