Talk 1 - Invariant measures and recurrence

Orbits under a transformation

For a set X endowed with a map $T : X \to X$, we denote the positive orbit of $x \in X$ as

$$O^+(x) = \{T^n x : n \in \mathbb{N}\}$$

And if T is supposed to be invertible, we denote the total orbit as:

$$O(x) = \{T^n x : n \in \mathbb{Z}\}$$

The first one corresponds to an action of $\mathbb{N}$ on X, the second corresponds to an action of $\mathbb{Z}$.

We also call flows the actions of $\mathbb{R}$ or $\mathbb{R}_+$, and define similarly the orbits.

The systems we care about

In this workshop, we care about two dynamical contexts:

Topological Dynamical Systems

That is the data of a *compact* topological space X, and of a continuous map $T : X \to X$.
Whenever T is a homeomorphism, we might talk of an invertible *tds*.

Probability Preserving Transformations:

That is the data of a probability space $(X, \mathcal{A}, \mu)$, and a probability preserving measurable function $T : X \to X$.
In other words, for any $A \in \mathcal{A}$, $T^{-1}(A) \in \mathcal{A}$, and we require

$$T_* \mu(A) = \mu(T^{-1}(A)) = \mu(A)$$

We always work in these compact contexts, that is with finite measure or compact topological spaces.

We might also encounter *measurable dynamical systems* when working without a given invariant measure, most notably, any topological dynamical system gives rise to a measurable dynamical system by considering the Borelian σ-algebra.

Examples:

The following examples will follow us for a while.

Rotations on the circle

Consider the topological space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the circle, and for an element $\alpha \in \mathbb{R}$, consider the action by $R_\alpha : x + \mathbb{Z} \mapsto x + \alpha + \mathbb{Z}$.
R_α preserves the Lebesgue measure on $\mathbb{T}$.

Action of a Lie group on a quotient space

If we consider G to be a Lie group (e.g. $\mathrm{SL}_2(\mathbb{R})$), with its Haar measure μ, and take $\Gamma < G$ to be a lattice of G, that is a discrete subgroup with a finite induced measure (e.g. $\mathrm{SL}_2(\mathbb{Z})$), we then find an invertible measure preserving transformation for any $h \in G$:

$$T_h = h \cdot - : G/\Gamma \to G/\Gamma$$

And in the specified case of a co-compact lattice Γ, this defines moreover a topological dynamical system in our sense.

Bernoulli shift

$X = \{0,1\}^{\mathbb{N}}$. $\sigma : (u_n) \mapsto (u_{n+1})$.
Choose $p = (p_0, p_1) \in \mathbb{R}_+^2$, such that $p_0 + p_1 = 1$.

We consider the base opens of the σ-algebra to be the cylinders given by a finite word $w \in \{0,1\}^k$:

$$C(w) = \{(x_i) \in X \mid \forall i \leq k,\, x_i = w_i\}$$

The measure of a cylinder is then set to be $\mu(C(w)) = \prod_i p_{w_i}$.

$\sigma_* \mu$ and μ agree on the cylinders, and as such, by Caratheodory theorem, both measures agree on the whole boolean σ-algebra, and we thus have a probability preserving transformation.

2-cover of the circle by itself

Consider $T_2 : \mathbb{T} \to \mathbb{T}$, $T_2(x + \mathbb{Z}) = 2x + \mathbb{Z}$.

This transformation preserves the Lebesgue measure, as the preimage of an interval $I \subset [0,1[\subseteq \mathbb{T}$ of length l is the union of two intervals of length $l/2$.

Baker's map

We consider the map $T : [0,1]^2 \to [0,1]^2$, $T(x,y) = (2x \mod 1, (y + [2x])/2)$
Where $[-] : \mathbb{R} \to \mathbb{Z}$ denotes the floor map.

T is measurable and preserves the Lebesgue measure.
T is invertible in a measured sense, that is we can define a map $S : [0,1]^2 \to [0,1]^2$ such that $T \circ S = S \circ T = \mathrm{id}_{[0,1]^2}$ for almost every point.

In terms of functionals and operators

Remember that measures on $(X, \mathcal{A})$ can dually be seen as a functional: the integral.
Most notably we care about measures of finite total weight, their vector space $\mathcal{M}(X, \mathcal{A})$ is the linear dual of $l^\infty(X, \mathcal{A})$, the space of bounded measurable functions from X to $\mathbb{R}$.

We can accordingly characterise functionally what it means for a measurable map $T : X \to X$ to preserve $\mu \in \mathcal{M}(X)$:

$$\mu \in \mathcal{M}(X)^T \iff T_* \mu = \mu \iff \forall f \in L^1(X, \mu), \int_X f \, d\mu = \int_X f \circ T \, d\mu$$

In particular, if T preserves μ, we can define the Koopman operator U_T, which is an isometry on each $L^p(X, \mu)$ as:

$$U_T : f \mapsto f \circ T$$

Periodicity, recurrence, minimality

Definitions

We consider a topological dynamical system $T : X \to X$.

We say that some element $x \in X$ is

- *Periodic* if there exists some $n \geq 1$ such that $T^n x = x$.
- *Pre-periodic* if its orbit is finite. (Or equivalently that its orbit contains a periodic element.)
  If (X, d) is metric:
- *Quasi-periodic* if $\forall \epsilon > 0$, $\{n \in \mathbb{N} \mid d(T^n x, x) < \epsilon\}$ has bounded gaps.

We say that $x \in X$ is

- *recurrent* if there exists an extraction $n_j \to \infty$ such that $T^{n_j} x \to x$.

It is clear that periodicity implies quasi-periodicity, which itself implies recurrence.

We say that the system (X, T) is *minimal* there is no proper closed subset $A \Subset X$ which is invariant under T: $TA \subseteq A$. Equivalently, (X, T) is minimal if all orbits are dense.

Examples

The rotation R_α on the circle is minimal for a number of turns α irrational.

Remark

The sequence of closest returns of R_α are the $(q_n)_n$ coming from the continuous fraction expansion of α:

$$\alpha = a_0 + 1/(a_1 + 1/(\ldots))$$

Where p_n/q_n is the truncated fraction.

For the Bernoulli shift (X, σ), the countable subset $P \subset X$ of periodic words is dense in X, and from that we can construct a dense orbit.

The following theorems are deeply tied to the compacity of our systems, may it be topological compacity, or finite total weight of measures.

Birkhoff recurrence theorem

Let (X, T) be a topological dynamical system, then it admits a recurrent point.

Poincaré recurrence theorem

Let $(X, \mathcal{A}, \mu, T)$ be a probability preserving transformation.
Let $A \in \mathcal{A}$, then μ-almost every $x \in A$ returns to A infinitely many times.
In other words, there exists $B \subseteq A$, such that $\mu(A) = \mu(B)$ and $\forall x \in B$, there exists $n_j \to \infty$ such that $T^{n_j}x \in A$.

Proof of Poincaré recurrence

Let us define

$$E_1 = \{x \in A \mid \forall n > 0, T^n x \notin A\}$$

And let us consider the different $T^{-n}E_1$, $n \in \mathbb{N}$.
Since T preserves measure, $\mu(T^{-n}E_1) = \mu(E_1)$.
By definition of E, $T^{-n}E \cap T^{-m}E = \emptyset$ whenever $n \neq m$.

Thus necessarily, E is of measure 0, since X is of **finite total measure**.

We have shown that almost every $x \in A$ comes back to A under T at some point.

The same thing is verified for each T^k, as we can similarly define E_k, also of measure 0, and if we set

$$B = A \setminus (E_1 \cup E_2 \cup \ldots) = \{x \in A \mid \forall k, \exists n > k,\ T^n x \in A\},$$

then B has the same measure as A.

Factorization of systems

Let (X, A, μ, T) and (Y, B, ν, S) be ppt.

(Y, S) is said to be a factor of (X, T) if there exists some stable $X' \subseteq X$, a stable $Y' \subseteq Y$ of total measure, and a measurable map $\phi : X' \to Y'$ which preserves the actions and measures:

$$\phi \circ S = T \circ \phi \quad ; \quad \phi_* \nu = \mu$$

Accordingly, we define isomorphisms when ϕ is bi-measurable, invertible, and ϕ^{-1} defines a factorization.

What category of measured space do we consider?

We consider here morphisms $f : (Y, \nu) \to (X, \mu)$ between measured space to be measure preserving measurable functions: $f_* \nu = \mu$, which are defined ν-almost everywhere, and up to the

equivalence relation of agreeing ν-almost everywhere.
This defines a category $\mathcal{M}$ of measured spaces (of finite total weight).

The induced category of measure-preserving-transformation is the functor category $\mathrm{Fun}(\mathrm{B}\mathbb{N}, \mathcal{M})$, and factorizations then correspond to these morphisms.

In the same way, we define a factorization of topological dynamical systems to be $\phi : (Y, S) \to (X, T)$ to be a continuous map $\phi : Y \to X$ which preserves the action:

$$\phi \circ S = T \circ \phi$$

Isomorphisms of systems are similarly defined.

Minimality and Factorization

It is clear from this definition that a minimal topological dynamical system corresponds to a system for which every factorisation is surjective (or empty).

Examples

The 1-sided bernoulli shift $(\{0,1\}^{\mathbb{N}}, \sigma)$ is a topological factor of the 2-sided Bernoulli shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$

The 2-cover of the circle is a factor of the Baker's map

The 2-cover of the circle is isomorphic to the $(1/2, 1/2)$-Bernoulli shift, by considering the binary expansion.
This is of course when considering them as measured systems, on the topological side, we only find that one is a factor of the other.