# LECTURE 5 — THEORY OF LOCAL NEWFORMS FOR $U(3)$

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Abstract. In this short note we will discuss Miyauchi's proof of existence and multiplicity one property of newforms for an irreducible admissible generic representation of the unramified unitary group in three variables U(2, 1) defined over a $p$-adic field.

Let $F$ be non-archimedean local of mixed characteristic $(0, p)$ with $p \neq 2$. Denote $\mathfrak{O}_F$ its ring of integers, $\varpi_F$ a uniformizer, $\mathfrak{p}_F = (\varpi_F)$ the maximal ideal, $k_F = \mathfrak{O}_F/\mathfrak{p}_F$ the residue field and $q$ its cardinality. We normalize the absolute value $|\cdot|_F$ such that $|\varpi_F|_F = \frac{1}{2}$.

Let $E = F[\sqrt{\varepsilon}]$ a unramified quadratic extension of $F$, where $\varepsilon \in \mathfrak{O}_F^\times \backslash (\mathfrak{O}_F^\times)^2$. Denote similarly $\mathfrak{O}_E$ its ring of integers, $\varpi_E = \varpi_F$ a uniformizer, $\mathfrak{p}_E = (\varpi_E)$ and $k_E = \mathfrak{O}_E/\mathfrak{p}_E$. Denote $\bar{\cdot}$ the action of the non-trivial element of $\mathrm{Gal}(E/F)$.

Consider the unitary group

$$G = \{g \in \mathrm{GL}_3(E) \ : \ {}^t\bar{g}Jg = J\} \quad \text{where} \quad J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \tag{1}$$

and this group is also the unitary group $U(2, 1)(F)$. Introduce the upper triangular Borel subgroup

$$B = \begin{pmatrix} \star & \star & \star \\ & \star & \star \\ & & \star \end{pmatrix} \cap G \tag{2}$$

and the associated torus

$$T = \begin{pmatrix} \star & & \\ & \star & \\ & & \star \end{pmatrix} \cap B. \tag{3}$$

The unipotent radical of $B$ is

$$U = \left\{ u(x, y) := \begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \ : \ x, y \in E, \ y + \bar{y} + x\bar{x} = 0 \right\} \tag{4}$$

and its opposite

$$\widehat{U} = \left\{ \begin{pmatrix} 1 & & \\ x & 1 & \\ y & -\bar{x} & 1 \end{pmatrix} \ : \ x, y \in E, \ y + \bar{y} + x\bar{x} = 0 \right\}. \tag{5}$$

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We can embed the smaller unitary group $H = U(1,1)(E/F)$ into $G$ by

$$H \simeq \left\{ \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix} \cap G \right\} \tag{6}$$

and define similarly $B_H = B \cap H$, $U_H = U \cap H$ and $T_H = T \cap H$. Let $\psi_E$ be a nontrivial additive character of $E$ with conductor $\mathfrak{O}_E$, and we extend it to $U$ by setting $\psi_E(u(x,y)) = \psi_E(x)$ for all $u(x,y) \in U$.

Let $(\pi, V)$ be an irreducible admissible representation of $G$. By multiplicty one, we know that $\dim \mathrm{Hom}_U(\pi, \psi) \leqslant 1$.

**Definition.** The representation $(\pi, V)$ is said to be *generic* if $\mathrm{Hom}_U(\pi, \psi) \neq \{0\}$ (i.e. $\dim = 1$).

For all $n \geqslant 0$, define

$$K_n = \begin{pmatrix} \mathfrak{O}_E & \mathfrak{O}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n & \mathfrak{O}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{O}_E \end{pmatrix} \cap G. \tag{7}$$

Then$K_0$ is a “good” maximal compact subgroup of $G$.

Let us define $V(n) := \{v \in V \ : \ \forall k \in K_n, \ \pi(k)v = v\}$ the $K_n$-fixed space in $V$. It is a finite-dimensional subspace of $V$ by admissibility.

**Theorem.** *If $(\pi, V)$ is generic, then there exists $n \in \mathbb{N}$ such that $V(n) \neq 0$.*

*Proof.*

**Lemma.** For all $n \in \mathbb{N}$, $K_n$ is generated by $K_n \cap H$ and $U(\mathfrak{O}_E)$.

*Proof.* □

□

**Definition.** Define the conductor of $(\pi, V)$ as

$$N_\pi = \min\{n \in \mathbb{N} \ : \ V(n) \neq 0\}. \tag{8}$$

The space $V(N_\pi)$ is called the space of *newforms* of $\pi$. For $n > N_\pi$ the subspace $V(n)$ is called a space of *oldforms*.

Let $Z$ be the center of $G$, and we have that $Z$ is isomorphic to $E^1$, the norm-one subgroup of $E^\times$. This isomorphism can be explicitly given as

$$\begin{aligned} \iota : Z &\longrightarrow E^1 \\ \lambda &\longmapsto \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}. \end{aligned} \tag{9}$$

Define $E_n^1 = E^1 \cap (1 + \mathfrak{p}_E^n)$ for $n \in \mathbb{N}$ which are open compact subgroups of $E^1$. We set $Z_n := \iota(E_n^1)$.

Let $\omega_\pi$ be the central character of $\pi$ and $n_\pi$ its conductor, i.e.

$$n_\pi = \min\{n \geqslant 0 \ : \ (\omega_\pi)_{|Z_n} = 1\}. \tag{10}$$

*Remark.* Since $Z_n = Z \cap K_n$, $\omega_\pi$ is trivial on $Z_n$ if $V(n) \neq 0$, so that $N_\pi \geqslant n_\pi$.

**Theorem.** *If $(\pi, V)$ is irreducible admissible and admits a newform, then* $\dim V(N_\pi) = 1$.

*Proof.* □

**Theorem.** *Let $(\pi, V)$ be an irreducible admissible representation of $G$ with $N_\pi > n_\pi$ and $N_\pi \geqslant 2$.*

(i) *If $n > n_\pi$, then* $\dim V(n) - \dim V(n-2) \leqslant 1$.
(ii) *If $N_\pi > n_\pi$, then* $\dim V(N_\pi) = 1$.

*Proof.* Note that (ii) is a consequence of (i).

Let us do some preparations in order to prove $(i)$.

**Level raising operators:** Introduce

$$\eta := \begin{pmatrix} \varpi^{-1} & & \\ & 1 & \\ & & \varpi \end{pmatrix} \tag{11}$$

so that $K_{n+2} \subset \eta K_n \eta^{-1}$ for each $n \geqslant 0$. We hence get a level raising operator

$$\begin{array}{cccc} \eta & : & V(n) & \longrightarrow & V(n+2) \\ & & v & \longmapsto & \pi(\eta)v. \end{array} \tag{12}$$

We also have another "naive" level raising operator obtained by simply averaging:

$$\begin{array}{cccc} \theta' & : & V(n) & \longrightarrow & V(n+1) \\ & & v & \longmapsto & \theta'(v) := \mathrm{vol}(K_{n+1} \cap K_n)^{-1} \int_{K_{n+1}} \pi(k) v \mathrm{d}k. \end{array} \tag{13}$$

*Remark.* Note that we have $\eta\theta' v = \theta' \eta v$ for any $v \in V(n)$, i.e. both operators commute on $V(n)$.

Lastly, we introduce another averaging operator. Let

$$U(\mathfrak{p}_E^{-1}) := \begin{pmatrix} 1 & \mathfrak{p}_E^{-1} & \mathfrak{p}_E^{-2} \\ & 1 & \mathfrak{p}_E^{-1} \\ & & 1 \end{pmatrix} \cap G \tag{14}$$

and define the operator $S$ on $V$ by

$$S : v \longmapsto \mathrm{vol}(U(\mathfrak{O}_E))^{-1} \int_{U(\mathfrak{p}_E^{-1})} \pi(u) v \mathrm{d}u. \tag{15}$$

**$P_2$-modules:** Introduce

$$P_2 := \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix} \subset GL_2(E). \tag{16}$$

We have an isomorphism

$$\begin{array}{ccc} T_H U / U_H & \xrightarrow{\sim} & P_2 \\ t(a) u(x,y) & \longmapsto & \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \end{array} \tag{17}$$

where $a \in E^\times$ and $x, y \in E$. Define

$$V(U) = \langle \pi(u)v - v \mid v \in V, u \in U \rangle \quad \text{and} \quad V_U := V/V(U), \tag{18}$$

$$V(U_H) = \langle \pi(u)v - v \mid v \in V, u \in U_H \rangle \quad \text{and} \quad V_{U_H} := V/V(U_H). \tag{19}$$

$V_{U_H}$ is a $P_2$-module and let $\mathrm{proj} : V \to V_{U_H}$ denote the projection. Let

$$C_2 := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \subset \mathrm{GL}_2(E) \tag{20}$$

corresponding to $Z_2$ in Miyauchi's paper. Then define

$$V_{U_H}(C_2) := \langle \pi(u)v - v \mid v \in V_{U_H}, u \in C_2 \rangle. \tag{21}$$

Then $V_{U_H}(C_2)$ is a $P_2$-submodule of $V_{U_H}$ and $V_{U_H}/V_{U_H}(C_2) \simeq V_U$. Moreover, $U/U_H \simeq C_2$ and $\mathrm{proj}(V(U)) = V_{U_H}(C_2)$.

**Lemma.** Let $(\pi, V)$ be an irreducible admissible representation of $G$ and $n \in \mathbb{N}$ such that $n \geqslant 2$ and $n > n_\pi$. Let $v \in V(n)$ such that $(S - q^4)v \in V(U_H)$ (i.e. $\mathrm{proj}((S - q^4)v) = 0$), then we must have $v \in \eta V(n-2)$.

In other words, this lemma provides a sufficient condition for a level $n$ vector $v$ to arise from the lower level $n - 2$.

**Kirillov model:** In this subsection we will assume that $(\pi, V)$ is generic. By Frobenius reciprocity, we have

$$\mathrm{Hom}_U(\pi, \psi) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \psi) \simeq \mathbf{C}. \tag{22}$$

So $\pi$ embeds uniquely into $\mathrm{Ind}_U^G \psi$ up to scalar. The image of $V$ denoted $\mathcal{W}(\pi, \psi)$ is called the *Whittaker model* of $\pi$. Let $\ell \in \mathrm{Hom}_U(\pi, \psi)$ be a nonzero Whittaker functional, and define the Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ associated to $v$ by

$$W_v(g) := \ell(\pi(g)v), \quad g \in G. \tag{23}$$

**Definition.** For $v \in V$, define a function on $E^\times$ as

$$\phi_v(a) := W_v \begin{pmatrix} a & & \\ & 1 & \\ & & \bar{a}^{-1} \end{pmatrix}, \quad a \in E^\times. \tag{24}$$

The function $\phi_v$ is in $C^\infty(E^\times)$, i.e. a locally constant function on $E^\times$.

We get a map

$$\begin{array}{ccc} V & \xrightarrow{v \mapsto \phi_v} & C^\infty(E^\times) \\ & {\scriptstyle \mathrm{proj}} \searrow \quad \nearrow {\scriptstyle v \mapsto \phi_v} & \\ & V_{U_H} & \end{array}$$

which factors through $V_{U_H}$. Define an action of $P_2$ on $C^\infty(E^\times)$ as

$$\left(\begin{pmatrix} a & x \\ & 1 \end{pmatrix} \phi\right)(b) := \psi_E(bx)\phi(ab), \qquad a, b \in E^\times, \ x \in E. \tag{25}$$

This turns the map $V_{U_H} \to C^\infty(E^\times)$ into a $P_2$-homomorphism.

**Lemma.** (i) Let $v \in V$. Then there exists $m \in \mathbb{N}$ such that $\mathrm{supp}(\phi_v) \subset \mathfrak{p}_E^{-m}$.
(ii) Let $v \in V(n)$. Then $\phi_v$ is $\mathfrak{O}_E^\times$-invariant and $\mathrm{supp}(\phi_v) \subset \mathfrak{O}_E$.

Let $C_c^\infty(E^\times) \subset C^\infty(E^\times)$ denote the subspace of compactly supported functions on $E^\times$.

**Proposition.** *Let $v \in V$ such that $\mathrm{proj}(v) \in V_{U_H}(C_2)$. Then $\phi_v \in C_c^\infty(E^\times)$.*

*Proof.* To do

**Lemma.** By the proposition we have a map $V_{U_H}(C_2) \to C_c^\infty(E^\times)$. This map is an isomorphism of $P_2$-modules.

**Lemma.** Let $n \geqslant 2$ such that $n > n_\pi$. Suppose $v \in V(n)$ is such that $\mathrm{supp}(\phi_v) \subset \mathfrak{p}_E$. Then $v \in \eta V(n-2)$.

*Proof.* The assumption implies that $\phi_{(S-q^4)v} = 0$. Since $U/U_H \simeq C_2$ we get that $\mathrm{proj}((S - q^4)v) \in V_{U_H}(C_2)$. By the above lemma, we get that $\mathrm{proj}((S - q^4)v) = 0$ i.e. $(S - q^4)v \in V(U_H)$. Hence, by the first lemma, we get that $v \in \eta V(n-2)$. $\square$

*Proof.* (of Theorem (i)) Since $\eta : V(n-2) \to V(n)$ is injective, it is enough to show that $V(n)/\eta V(n-2) \leqslant 1$. Let $v_1, v_2 \in V(n) \backslash \eta V(n-2)$. By the second lemma (ii), $\phi_{v_i}$ is $\mathfrak{O}_E^\times$-invariant and $\mathrm{supp}(\phi_{v_i}) \subset \mathfrak{O}_E$. Let $\alpha = \phi_{v_1}(1)$ and $\beta = \phi_{v_2}(1)$. By the last lemma, we have $\alpha \neq 0$ and $\beta \neq 0$.

But $\mathrm{supp}(\alpha v_1 - \beta v_2) \subset \mathfrak{p}_E$ so, again by the last lemma, we obtain that $\alpha v_1 - \beta v_2 \in \eta V(n-2)$. Hence $\dim V(n)/\eta V(n-2) \leq 1$. □

**Corollary.** *For $v \in V(N_\pi)$ nonzero, we have $W_v(1) \neq 0$.*

*Proof.* $\phi_v$ is supported on $\mathfrak{O}_E$. Since $V(N_\pi - 2) = 0$, we have that $\mathrm{supp}(\phi_v) \not\subset \mathfrak{p}_E$. Since $\phi_v$ is $\mathfrak{O}_E^\times$-invariant, we get that $\phi_v(1) = W_v(1) \neq 0$. □

Finally, we give a description of space of oldforms.

**Theorem.** *Let $(\pi, V)$ be an irreducible admissible generic representation of $G$. Then $W_v(1) \neq 0$ for $v \in V(N_\pi) \backslash \{0\}$. For $n \geqslant N_\pi$, the set*

$$\{(\theta')^i \eta^j v \ : \ i + 2j + N_\pi = n\} \tag{26}$$

*forms a basis of $V(n)$. In particular*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1. \tag{27}$$

## References


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