LECTURE 4 — THEORY OF LOCAL NEWFORMS FOR GL(n)

TALK BY CHI-YUN HSU

Abstract. We present the main ingredients of the proof of Jacquet, Piatetski-Shapiro and Shalika main result and its consequences.

1. Setting

Consider F a non-archimedean local field, $\psi : F \to \mathbf{C}^\times$ a nontrivial character of conductor $\mathfrak{O}$, and (π, V) an irreducible admissible representation of $G_r := \mathrm{GL}_r(F)$. To this representation, we can associate an L-factor of the form

$$L(s,\pi) = P_\pi(q^{-s})^{-1}, \quad P_\pi \in \mathbf{C}[X],\ P_\pi(0) = 1, \tag{1}$$

which has originally been constructed from Godement-Jacquet theory. It satisfies a functional equation involving an epsilon factor

$$\varepsilon(s,\pi,\psi) = Cq^{-ms},\ m \in \mathbf{Z}. \tag{2}$$

Recall the definition of the congruence subgroups $K(n)$ where

$$K(n) = \begin{pmatrix} \mathrm{GL}_{r-1}(\mathfrak{O}) & \mathfrak{O} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{pmatrix}, \qquad n \in \mathbb{N}, \tag{3}$$

and its subgroup

$$K'(n) = \begin{pmatrix} \mathrm{GL}_{r-1}(\mathfrak{O}) & \mathfrak{p}^n \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbf{Z}. \tag{4}$$

Each of them contains the compact subgroup

$$K = \begin{pmatrix} \mathrm{GL}_{r-1}(\mathfrak{O}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The main theorem of Jacquet, Piatetski-Shapiro and Shalika is the following.

Theorem (1). *Assume π is generic. Then*

- $m \geqslant 0$
- *if* $\pi^{K(n)} \neq 0$, *then* $n \geqslant m$
- $\dim \pi^{K(m)} = 1$

Date: April 14, 2022.
Notes updated on April 20, 2022.
These notes have been taken on the go by Didier Lesesvre. They may contain typos and errors.

2. Rankin-Selberg zeta integrals

2.1. **Zeta integrals.** Recall some facts about Rankin-Selberg L-functions, that have been introduced and extensively developped in [5]. Let $r' < r$, and consider (π', V') be an irreducible admissible represenation of $\mathrm{GL}_{r'}(F)$. Assume π and π' are generic. Given Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \bar{\psi})$. Define the Rankin-Selberg zeta integral

$$Z(s, W, W') = \int_{N_{r'} \backslash G_{r'}} W \begin{pmatrix} h & \\ & I \end{pmatrix} W'(h)|h|^{s-\frac{1}{2}} \mathrm{d}h, \tag{6}$$

where $|h|$ stands for $|\det(h)|$. We can write it as a formal series (with both positive and negative degrees) in q^{-s} by cutting by valuation of $\det(h)$, i.e.

$$Z(s, W, W') = \sum_{n \in \mathbf{Z}} a_n (q^{-s})^n \in \mathbf{C}[[q^{\pm s}]] \tag{7}$$

where

$$a_n = q^{n/2} \int_{val(|g|)=n} W \begin{pmatrix} h & \\ & I \end{pmatrix} W'(h)|h|^{-1/2} \mathrm{d}h. \tag{8}$$

This series has in fact only finitely many negative terms, so that

$$Z(s, W, W') = \sum_{n \in \mathbf{Z}} a_n X^n \in \mathbf{C}((q^{-s})) \tag{9}$$

by arguments on the support of W (see previous talk).

2.2. **Unramified principal series.** Choose π' to be an unramified principal series $\pi' = I(x_1, \dots, x_{r'})$ where $(x_1, \dots, x_{r'})$ are the Satake parameters (we can write it as an explicit induced representation).

In the Whittaker model $\mathcal{W}(\pi', \bar{\psi})$, there is a unique normalized spherical vector

$$W_{\underline{x}} = W'_\circ(\cdot; x_1, \dots, x_{r'}).$$

We in fact can show (it is a result of explicit formulas of Shintani for the unramified case for GL(n), and of Casselman-Shalika for a quite more general class of groups) that for all $g' \in G_{r'}$,

$$W'_\circ(g'; x_1, \dots, x_{r'}) \in \mathbf{C}[x_1^{\pm 1}, \dots, x_{r'}^{\pm 1}]^{\mathfrak{S}_{r'}}. \tag{10}$$

By the Satake isomorphism, this ring is isomorphic to the spherical Hecke algebra $\mathcal{H}_{r'}^{\mathrm{sph}}$. When W' is chosen to be $W_\circ(\underline{x})$ in the formula (7) for $Z(s, W, W')$, then it can be seen as a polynomial

$$\Psi_W(X, X_1, \dots, X_{r'}) \in \mathbf{C}[X_1^{\pm}, \dots, X_{r'}^{\pm}]^{\mathfrak{S}_{r'}} \tag{11}$$

such that

$$\Psi_W(q^{-s}, x_1, \dots, x_{r'}) = Z(s, W, W_{\underline{x}}). \tag{12}$$

2.3. **Main theorem.** We can now state the second main theorem of Jacquet, Piatetski-Shapiro and Shalika.

Theorem (2)**.** *Let $r' = r - 1$. There is a unique $W_\pi \in \mathcal{W}(\pi, \psi)$ such that*

- *W_π is right-K-invariant*
- $\Psi_{W_\pi}(X, X_1, \dots, X_{r'}) = \prod_{i=1}^{r'} P_\pi^{-1}(X X_i)$

In particular, taking $X = q^{-s}$ and $X_i = x_i$ we obtain

$$Z(s, W_\pi, W_{\underline{x}}) = L(s, \pi \times I(x_1, \dots, x_{r'})) \tag{13}$$

since the L-factor on the right is the L-factor of $L(s, \pi \times I(x_1, \dots, x_{r'})$ — and happens to also be the product of the local L-factors $L(s, \pi \times \chi_{x_i}) = L(s + x_i, \pi)$. In other words, the L-factor is attained by a zeta integral, and the *newvector* W_π is a test vector realising the L-factor as such a zeta integral.

2.4. **Three properties.** Let's now recall some important ingredients that will be crucial to the proofs.

Proposition (Support condition)**.** *There is a constant $C \geqslant 0$ such that if g is in the support of W, i.e. if $W(g) \neq 0$, then $\left|\frac{a_i(g)}{a_{i+1}(g)}\right| \leqslant C$ for all $i \leqslant r-1$, where we wrote $g = nak$ according to the Iwasawa decomposition and $a_i(g)$ is the i-th entry of the diagonal component a. Moreover, if W is fixed by $K'(0)$ (in fact, it is sufficient to take any subgroup containing matrice with integral nonzero $n_{i,i+1}$ coefficients), then we can take $C = 1$, in which case we have*

$$\Psi_W(X, X_1, \dots, X_{r'}) \in \mathbf{C}[X, X_1, \dots, X_r]. \tag{14}$$

Proof. See previous talk, and use critically the fact that the function in the zeta integral is of the form $W\left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right)$ so that $\alpha_r(g) = a_r(g) = 1$. □

Proposition (Functional equation for Ψ)**.** *We have the functional equation*

$$\Psi_W(X, X_1, \dots, X_{r'}) \prod_i P_\pi(XX_i) \prod_i \varepsilon(XX_i, \psi) \tag{15}$$
$$= \Psi_{\tilde{W}}(q^{-1}X^{-1}, X_1^{-1}, \dots, X_{r'}^{-1}) \prod_i P_{\tilde{\pi}}(q^{-1}X^{-1}X_i^{-1}) \tag{16}$$

where $\tilde{W}(g) = W(w^t g^{-1})$ is a Whittaker function for the contragredient $\tilde{\pi}$.

Proof. This is proven in [5]. □

Introducing the notation

$$\Phi_W(X, X_1, \dots, X_{r'}) = \Psi_W(X, X_1, \dots, X_{r'}) \prod_i P_\pi(XX_i) \tag{17}$$

and $\varepsilon(X, \underline{X}) = \prod_i \varepsilon_\pi(XX_i)$, this rewrites more compactly as

$$\Phi_W(X, \underline{X}) = \varepsilon(X, \underline{X}) \Phi_{\tilde{W}}(q^{-1}X^{-1}, \underline{X}^{-1}). \tag{18}$$

Note also that $\Phi_W(X, X_1, \dots, X_{r'})$ can be thought of as the local zeta integral $\Psi_W(X, \underline{X})$ from which we removed the common denominator.

Proposition (Injectivity of zeta integrals)**.** *If $\Psi_W(X, X_1, \dots, X_{r-1}) = 0$, then $W = 0$.*

Proof. If $\Psi_W(X, X_1, \dots, X_{r-1}) = 0$, then $W\left(\begin{smallmatrix} h & \\ & 1\end{smallmatrix}\right) = 0$ for all $g \in G_{r-1}$, by [4]. In fact, this implies that $W = 0$ as a consequence of Bernstein-Zelevinsky results. □

3. Proof of Theorem 1 from Theorem 2

Let's show that the theorem 2 implies theorem 1. So assume from now on that the theorem 2 is valid, and in particular there exists such a newvector W_π.

Proposition. *If Theorem 2 holds, then W_π is fixed by $K(m)$ and $m \geqslant 0$.*

Remark. We know, by smoothness, that it is fixed by a certain $K(n)$. The whole point of Theorem 1 is to prove that this n can be taken to be the same integer m appearing as exponent of the ε-factor.

Proof. Note that we have $K(m) = \langle {}^tK'(m), K'(0)\rangle$ for $m \geqslant 0$, so it is enough to prove the invariant by both $K'(0)$ and ${}^tK'(m)$ separately.

Step 1. W_π is fixed by $K'(0)$.

It is the same as saying that the support of W has its last lines integral. This is proven in the paper of Jacquet, Piatetski-Shapiro and Shalika.

Step 2. W_π is fixed by $K(m)$.

By the functional equation for Ψ_{W_π}, changing variables, using the fact that $\varepsilon(X) = CX^m$ and the newform definition "$\Psi_{W_\pi} P_\pi = 1$", we get

$$\Psi_{\tilde{W_\pi}}(X, X_1, \dots, X_{r-1}) = \prod_i P_{\tilde{\pi}}^{-1}(XX_i) \prod_i C(q^{-1}X^{-1}X_i^{-1})^m. \tag{19}$$

We will construct another Whittaker function having the same Ψ-integral, so that it will be equal to $\tilde{W_\pi}$ by injectivity (note that $\tilde{W_\pi}$ may not be the newvector $W_{\tilde{\pi}}$ for $\tilde{\pi}$). Introduce

$$W_m := \begin{pmatrix} \varpi^m I_{r-1} & \\ & 1 \end{pmatrix} \cdot W_{\tilde{\pi}} \tag{20}$$

Then, changing variables and using the action of the central character,

$$\Psi_{W_m}(X, X_1, \dots, X_{r-1}) = \Psi_{W_{\tilde{\pi}}}(X, X_1, \dots, X_{r-1}) \prod_i X_i^{-m} (Xq^{1/2})^{-m(r-1)} \tag{21}$$

where $\prod x_i^{-m}$ is the value of the central character of the induced $I(\cdot, x_1, \dots, x_{r-1})$ on the scalar matrix $\varpi^{-m} I_{r-1}$, and the last term comes from $|g|^{s-1/2}$ after the change of variables. Now, by the main theorem, this is also

$$\prod_i P_{\tilde{\pi}}^{-1}(XX_i)(Xq^{1/2})^{-m(r-1)} \prod_i X_i^{-m} \tag{22}$$

hence $\Psi_{\tilde{W_\pi}} = \alpha \Psi_{W_m}$ where α is a constant. By the injectivity property, we deduce $\tilde{W_\pi} = \alpha W_m$. But since $W_{\tilde{\pi}}$ is fixed by $K'(0)$ by the previous step, W_m is fixed by the conjugate of $K'(0)$ by ϖ^m, which is $K'(m)$. Thus $\tilde{W_\pi}$ is fixed by $K'(m)$, i.e. W_π is fixed by ${}^tK'(m)$ too. If $m \geqslant 0$, both subgroups generate $K(m)$. If $m < 0$, then they generate G_r which is absurd (see e.g. the equality defining the newvector). □

Now let's prove the two last properties of the Theorem 1.

Proposition. *If Theorem 2 holds, then*

- $\dim \pi^{K(m)} = 1$
- *if* $\pi^{K(n)} \neq 0$ *then* $n \geqslant m$

Proof. Let $W \in \mathcal{W}(\pi, \psi)$ fixed by $K(n)$. It in particular is fixed by $K'(0)$ and ${}^tK'(n)$, so that $\tilde{W}$ is fixed by $K'(n)$. Then $\tilde{W}_{-n} := \varpi^{-n} \cdot \tilde{W}$ is fixed by $K'(0)$.

By similar computations as above, we obtain

$$\Psi_{\tilde{W}_{-n}}(X, \underline{X}) = \Psi_{\tilde{W}}(X, \underline{X})(Xq^{1/2})^{n(r-1)} \prod_i X_i^n \tag{23}$$

so, multilying by $\prod_i P_{\tilde{\pi}}(XX_i)$, we obtain

$$\Psi_{\tilde{W}_{-n}}(X,\underline{X})\prod_i P_{\tilde{\pi}}(XX_i) = \Psi_{\tilde{W}}(X,\underline{X})\prod_i P_{\tilde{\pi}}(XX_i)\cdot(Xq^{1/2})^{n(r-1)}\prod_i X_i^n. \tag{24}$$

Using the functional equation and the fact that $\varepsilon(X) = CX^m$, this also rewrites

$$= \Psi_W(q^{-1}X^{-1}, q^{-1}X_i^{-1})\prod_i P_\pi(q^{-1}X^{-1}X_i^{-1})\prod_i C(q^{-1}X^{-1}X_i^{-1})^m(Xq^{1/2})^{n(r-1)}\prod_i X_i^n. \tag{25}$$

By the $K'(0)$-invariance of W (resp. $\tilde{W}_{-n}$), $\Psi_W(X,\underline{X})\prod P_\pi(XX_i)$ (resp. the corresponding quantity for $\tilde{W}_{-n}$) is a polynomial in the X_i. Looking at the X_i-part of the equality between (24) and (25), we deduce from the factor $\prod(XX_i)^{n-m}$ that we need $n-m \geqslant 0$.

In the case $n = m$, we deduce from the above relation that Ψ_W is equal to Ψ_{W_π} up to a constant hence, by the injectivity property, we conclude that $W = W_\pi$ and this ends the proof of the multiplicity one result. □

4. An outlook of the proof of Theorem 2

4.1. **Uniqueness.** The uniqueness (up to a constant) of the newform is a consequence of the injectivity of the local zeta integrals: it determines W on $G_{r-1}\times 1$, and this determines W on the whole G_r (as a consequence of Bernstein-Zelevinsky results).

4.2. **Existence.** Let $W \in \mathcal{W}(\pi,\psi)$. Introduce

$$\Xi_W(XX_i) := \Psi_W(X,\underline{X})\prod_i P_\pi(XX_i) \tag{26}$$

where $\Xi_W \in \mathbf{C}[X_1^\pm,\dots,X_r^\pm]^{\mathfrak{S}_{r-1}}$. This is an easy consequence of the "homogeneity property" of Whittaker functions (and thus of zeta integrals Ψ_W), as detailed in [3]. Introduce the set of all such polynomials

$$I(\pi) := \{\Xi_W \ : \ W \in \mathcal{W}(\pi,\psi)\}. \tag{27}$$

Proposition. *$I(\pi)$ is a $\mathbf{C}[X_i]$-ideal.*

Proof. Multiplying by a polynomial can be translated by a Hecke action, through the Satake isomorphism

$$\begin{array}{ccc} \mathcal{H}^{\mathrm{sph}} \longleftrightarrow & \mathbf{C}[X_1^\pm,\dots,X_{r-1}^\pm]^{\mathfrak{S}_{r-1}} & \\ \phi & \longleftrightarrow & P \end{array} \tag{28}$$

In other words, the Hecke eigenfunction of ϕ is given by P, that is to say

$$\phi\star W'_\circ(g,\underline{x}) = P(\underline{x})W'_\circ(g,\underline{x}). \tag{29}$$

Defining

$$W_\phi(g) = \int_{G_r} W\left(g\begin{pmatrix} h & \\ & 1\end{pmatrix}\right)\phi(h^{-1})|h|^{-1/2}\mathrm{d}h, \tag{30}$$

we verify that $P\Xi_W = \Xi_{W_\phi}$, so that $I(\pi)$ is indeed an ideal. □

Proposition. *$I(\pi)$ is the whole ring $\mathbf{C}[X_1^\pm,\dots,X_{r-1}^\pm]^{\mathfrak{S}_{r-1}}$.*

Proof. This is a consequence of the huge paper [5], proving all the properties of Rankin-Selberg convolutions. □

Remark. This is where the paper [4] contains a (quite critical!) mistake. They show that two polynomials of the form $P_\pi(X)$ and $P_{\tilde{\pi}}(q^{-1}X^{-1})$ are in $I(\pi)$, and that these ideals are coprime, concluding the proof. However, they are not always coprime for $n \geqslant 3$ as noticed by Nadir Matringe. This was fixed in [3] (and Matringe proved the result using derivatives of representations).

Hence there is a $W_\pi \in \mathcal{W}(\pi, \psi)$ satisfying $\Xi_W = 1$, which is the desired statement. It remains to make it $K \times 1$ invariant, what is done by averaging (i.e. taking the convolution with $\mathbf{1}_{K\times 1}$. □

Remark. The $K \times 1$-invariance is not necessary to the proof, but is necessary to state uniqueness and to prove the finer properties proved above.

Remark. An analogous result for Godement Jacquet integrals is in Humprhies: there are (explicit) test vectors and matrix coefficients so that the associated Godement-Jacquet integral exactly matches the L-factor. However there is not such nice consequences as being fixed by certain subgroups; see Jana-Nelson for such statements.

References

[1] Cogdell, *Lectures on L-functions, Converse Theorems and Functoriality for* GL(n), in Lectures on Automorphic L-functions, AMS (2000).
[2] Godement-Jacquet, *Zeta Functions of Simple Algebras*, Lecture Notes in Math 260 (1972), Springer.
[3] Jacquet, *A correction to Conducteurs des représentations du groupe linéaire*, 2011.
[4] Jacquet, Piatetski-Shapiro and Shalika, *Conducteurs des représentations du groupe linéaire*, Math. Annalen 256 (1981), 199–214.
[5] Jacquet, Piatetski-Shapiro and Shalika, *Rankin-Selberg convolutions*, Amer. J of Math. 105 (1983), 367–464.
[6] Tsai, *On Newforms for Split Special odd orthogonal groups*, PhD Thesis at Harvard University (2013).