LECTURE 3 — THEORY OF LOCAL NEWFORMS FOR GL(n)

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Abstract. I will introduce some tools and ideas at the heart of the paper of Jacquet-Piatetski-Shapiro and Shalika, establishing the theory of local newforms on GL(n). The good properties of Whittaker functions make them have a "Fourier transform" that is polynomial in certain parameters. The existence (and other properties) of newforms can then be rephrased as properties on polynomials, and this point of view is the key of the proofs.

1. Landscape and L-factors

1.1. **Setting.** Let F be a non-Archimedean local field, $\mathfrak{O}$ its ring of integers. Let ψ be a nontrivial additive character of F of conductor $\mathfrak{O}$. Let $G_r = \mathrm{GL}_r(F)$ and consider π and irreducible admissible generic representation of G_r.

1.2. **L-factors.** We define L-factors from suitable zeta integrals. In the case of the $(\mathrm{GL}_r, \mathrm{GL}_{r-1})$ strong Gelfand pair, for π (resp. π') an irreducible generic representation of G_r (resp. G_{r-1}) and for Whittaker functions $W \in \mathcal{W}(\pi)$ (resp. $W' \in \mathcal{W}(\pi')$), introduce the (Rankin-Selberg) zeta integral

$$Z(s, W, W') = \int_{N\backslash G} W \begin{pmatrix} g & \\ & 1 \end{pmatrix} W'(g)|g|^{s-1/2}\mathrm{d}g. \tag{1}$$

Remark. Recall that this definition comes from the global zeta integrals $Z(s, \phi, \phi')$ on $G(F)\backslash G(\mathbf{A})$, that factorize as Euler products into integrals on $N(F)\backslash G(F)$, after using the Fourier expansion of the automorphic form ϕ.

Remark. More generally, if we are not in the case of $(\mathrm{GL}_r, \mathrm{GL}_{r-1})$, but of $(\mathrm{GL}_r, \mathrm{GL}_k)$ for $k < r-1$, a similar definition holds with the first factor that has to be projected on the smaller group by an integration process, see Cogdell's notes and Duc Nam's talk in the previous semester. The whole theory holds similarly in this case.

The Godement-Jacquet theory leads to the following properties, as a consequence of the properties of the Whittaker functions W and W':

Proposition. *We have the following:*

- *the zeta integrals $\psi(s, W, W')$ converges for $\Re(s) \geqslant 1$*
- *$\psi(s, W, W') \in \mathbf{C}(q^{-s})$, in particular extends meromorphically to all $\mathbf{C}$*
- *the vector subspace $I(\pi \times \pi')$ of $\mathbf{C}(q^{-s})$ generated by the $\psi(s, W, W')$, for $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$, is a $\mathbf{C}[q^s, q^{-s}]$-fractional ideal*
- *the ideal $I(\pi \times \pi')$ admits a generator of the form $P_{\pi\times\pi'}(q^{-s})$ where $P_{\pi\times\pi'} \in \mathbf{C}[X]$ is so that $P_{\pi\times\pi'}(0) = 1$*

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This motivates to consider:

Definition. We denote $L(s, \pi \times \pi') = P_{\pi\times\pi'}(q^{-s})^{-1}$ the Rankin-Selberg L-factor attached to $\pi \times \pi'$. We define $L(s, \pi) = L(s, \pi \times \chi_0)$ where χ_0 is the trivial character.

The L-factor $L(s, \pi)$ is henceforth the "common denominator" of all the elements in the ideal $\langle \psi(s, W) \rangle$. Being an element of the vector space $I(\pi)$, we only know that there is a finite sum such that

$$L(s, \pi) = \sum_i \psi(s, W_i). \tag{2}$$

One of the fundamental questions we are addressing here is the one of the existence of $W \in \mathcal{W}(\pi)$ such that $L(s, \pi) = \psi(s, W)$.

1.3. **Conductors.** Another fundamental property of the L-factors (as a consuence of the analogous property on zeta integrals) is the functional equation

$$L(s, \pi) = \varepsilon_\pi(s, \pi, \psi) L(1 - s, \tilde{\pi}) \tag{3}$$

where the *epislon factor* is in fact a monomial

$$\varepsilon_\pi(s, \pi, \psi) = C q^{-ms} \tag{4}$$

for a certain constant C (depending on ψ) and $m \in \mathbf{Z}$ (independent of ψ).

Example. (The case $r = 1$) If $r = 1$, then π is a character of $F^\times$ and it is known that $r \geqslant 0$. For $m = 0$, we say π is unramified and it is trivial on $\mathfrak{O}$. For $m > 0$, it is ramified and it is trivial on $1 + \mathfrak{p}^n$ for a certain $n \geqslant 1$, but not on $1 + \mathfrak{p}^{n-1}$. We can prove that $m = n$ in (4), so that the notion of conductor (coming from the functional equation) matches the notion of depth (coming from the filtration $(1 + \mathfrak{p}^n)_n$).

It is natural to wonder about similar interpretation of the conductor for $r = 2$. The answer is positive for $r = 2$ due to the work of Casselman, see Julien's talk. It also holds for $r \geqslant 2$ by Jacquet, Piatetski-Shapiro and Shalika's result we are exploring here. Introduce the congruence subgroups

$$K_0(m) := \begin{pmatrix} K_{r-1} & (\mathfrak{O}^\times)^{r-1} \\ (\mathfrak{p}^m)^{r-1} & 1 + \mathfrak{p}^m \end{pmatrix} \tag{5}$$

where $(\mathfrak{O}^\times)^{r-1}$ denotes the set of columns with $r - 1$ entries in $\mathfrak{O}^\times$, and $(\mathfrak{p}^m)^{r-1}$ denotes the set of lines with $(r - 1)$ entries in $\mathfrak{p}^m$.

Theorem (Jacquet, Piatetski-Shapiro and Shalika)**.** *We have the following properties:*

- $m \geqslant 0$
- *there is* $n \neq 0$ *such that* $\pi^{K_0(n)} \neq 0$*; denote by* n_π *the minimal such* n
- $\dim \pi^{K_0(n_\pi)} = 1$*; denote by* W *a nonzero fixed vector*
- *for all* $i \geqslant 0$, $\dim \pi^{K_0(n_\pi + i)} = i + 1$ *(this is a result due to Reeder)*
- $\psi(s, W) = L(s, \pi)$ *(in a loose sense, see below for a precise statement)*

Remark. In fact, JPSS main theorem is the existence and uniqueness of a W satisfying the last equality, and that is $K_{r-1} \times 1$-right-invariant, see below. The other properties, in particular the invariance by the congruence subgroups $K_0(n)$ and multiplicity one, are consequences of this definition.

2. Whittaker functions

2.1. **Setting.** A fundamental question will be to understand the zeta integrals $\psi(s, W, W')$, and their properties will be obtained as easy consequences of the properties of Whittaker functions $W \in \mathcal{W}(\pi)$ that we study in this section.

Recall the Iwasawa decomposition $G = NAK$ where A is the diagonal subgroup, N the upper unipotent subgroup, and $K = \mathrm{GL}_r(\mathfrak{O})$ the maximal compact open subgroup. For a given (nontrivial) additive character ψ of F, define the character θ of N by

$$\theta(n) = \prod_{i=1}^{r-1} \psi(n_{i,i+1}). \tag{6}$$

2.2. **Whittaker model.** Recall that π is said to be generic if $V_\pi \simeq \mathcal{W}$ a space of left-θ-invariant functions, i.e. such that

$$\mathcal{W}(ng) = \theta(n)W(g) \tag{7}$$

for all $n \in N$, $g \in G$ and $W \in \mathcal{W}$. In other words, π is generic if $\mathrm{Hom}_N(\pi_{|N}, \theta) \neq 0$, or again (by Frobenius reciprocity) of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G(\theta))$.

For a generic π, the space $\mathcal{W}$ is unique up to isomorphism, it does not depend on ψ, is denoted $\mathcal{W}(\pi)$ and called the Whittaker model of π. If π is generic, then so is its contragredient $\tilde{\pi}$, and the Whittaker functions of π are described by $W(w^t g^{-1})$ where $W \in \mathcal{W}(\pi)$.

2.3. **Support.** The Whittaker functions have pretty explicit properties and strong invariances that will ultimately be critical for the proof of JPSS result. We make these properties explicit here, providing detailed proofs or references.

By definition, Whittaker functions are defined by the relation

$$W(ng) = \theta(n)W(g) \tag{8}$$

and by smoothness, i.e. existence of a neighborhood of the identity fixing W on the right. Since this neighborhood can be taken to be a compact open subgroup with finite index in K, we are reduced (up to a finite sum) to understand the properties of W on A.

Proposition (Support)**.** *There is a constant $C \geqslant 0$ such that, if $W(g) \neq 0$ with the Iwasawa decomposition $g = nak$, then*

$$\left| \frac{a_i}{a_{i+1}} \right| \leqslant C \tag{9}$$

for all $i \leqslant r - 1$.

Proof. Indeed, compute

$$W(an) = W(ana^{-1}a) = \theta(ana^{-1})W(a) = W(a) \prod_{i=1}^{r-1} \psi\left(\frac{a_i}{a_{i+1}} n_{i,i+1} \right). \tag{10}$$

For small enough entries in n, it is close enough to the identity matrix so that, by smoothness of W, we have $W(an) = W(a)$. This implies that the product of characters on the right is trivial, hence that each $a_i a_{i+1}^{-1} n_{i,i+1}$ is in the conductor $\mathfrak{O}$ of ψ. Depending on the "level of smoothness", i.e. on the $n_{i,i+1}$ we can chose so that $W(an) = W(a)$, we obtain the desired result. □

2.4. **Unramified principal series.** Introduce a pretty elementary kind of induced representations on G_{r-1}: the spherical (or: unramified) principal series. We will repeatedly use the Whittaker functions attached to it in the Rankin-Selberg zeta integrals we are considering. Let $(x_1, \ldots, x_r) \in \mathbf{C}^r$ be such that $x_i \neq 0$ (and maybe some extra properties), and consider the induced representation $I(x_1, \ldots, x_r)$ from the character of the Borel defined by

$$\begin{pmatrix} a_1 & \star & \star & \star \\ & a_2 & \star & \star \\ & & \ddots & \star \\ & & & a_r \end{pmatrix} \longmapsto x_1^{v(a_1)} x_2^{v(a_2)} \cdots x_r^{v(a_r)}. \tag{11}$$

Defining the unramified character χ_i by $\chi_i(\varpi) = x_i$, the above character is $\chi_1 \otimes \cdots \otimes \chi_r$. The induced $I(x_1, \ldots, x_r)$ is a spherical representation, i.e. contains a "unique" K-fixed vector, which we denote by f and assume it is normalized so that $f(I) = 1$.

Denote by $H = \mathcal{H}(G//K)$ the Hecke algebra of compactly supported functions on G that are bi-K-invariant. For $\phi \in H$, the average

$$\int_{G/K} f(gh)\phi(h)\mathrm{d}h \tag{12}$$

is also K-invariant, and belongs to $I(x_1, \ldots, x_r)$. Hence there is a scalar $\lambda(\phi)$ such that

$$\int_{G/K} f(gh)\phi(h)\mathrm{d}h = \lambda(\phi) f. \tag{13}$$

It is morevoer straightforward that $\lambda : H \to \mathbf{C}$ is a homomorphism of the Hecke algebra, that depends on $\underline{x} = (x_1, \ldots, x_r)$ and may be denoted $\lambda_{\underline{x}}$.

2.5. **The Whittaker function associated to $\underline{x}$.** To each homomorphism $\lambda_{\underline{x}} : H \to \mathbf{C}$, there is a unique "formal Whittaker function" on G such that

- $W(1) = 1$
- $W(ng) = \theta(n) W(g)$ for all $n \in N$
- $\int_G \phi(h) W(gh)\mathrm{d}h = \lambda_{\underline{x}}(\phi) W(g)$

This is a property stated in Shintani, however it is not proved there. We denote by $W_{\underline{x}}(g)$ or $W(g, x_i)$ the corresponding function.

Introduce the application

$$\check{f}(g) = f\left(w^t g^{-1} w^{-1}\right). \tag{14}$$

Proposition (Inverse-transpose)**.** *We have*

$$\lambda_{\underline{x}}(\check{\phi}) = \lambda_{\underline{x}^{-1}}(\phi). \tag{15}$$

Proof. To obtain the value of $\lambda(\check{\phi})$, note that the character $\chi_1 \otimes \cdots \otimes \chi_r$ acts by

$$w \;{}^{\mathrm{t}}\!\begin{pmatrix} a_1 & \star & \star \\ & \ddots & \star \\ & & a_r \end{pmatrix}^{-1} w^{-1} = \begin{pmatrix} a_1^{-1} & \star & \star \\ & \ddots & \star \\ & & a_r^{-1} \end{pmatrix} \longmapsto (x_1^{-1})^{v(a_1)} (x_2^{-1})^{v(a_2)} \cdots (x_r^{-1})^{v(a_r)}.$$

This in particular implies that $\check{f}_{\underline{x}} = f_{\underline{x}}(w^t \cdot^{-1} w^{-1}) \in I(\underline{x}^{-1})$. Hence, changing variables and using the fact that $\check{f}_{\underline{x}}(I) = f_{\underline{x}}(I) = 1$,

$$\lambda_{\underline{x}}(\check{\phi}) = \int_G \check{\phi} f_{\underline{x}} = \int_G \phi \check{f}_{\underline{x}} = \lambda_{\underline{x}^{-1}}(\phi) \tag{16}$$

and we obtain the desired result. $\square$

Proposition (Homogeneity). *We have, for all $g \in G$,*

$$\lambda_{q^{-s}\underline{x}}(\phi) f_{q^{-s}\underline{x}}(g) = \lambda_{\underline{x}}(\phi)|g|^s f_{\underline{x}}(g). \tag{17}$$

Multiplying by q^{-s} each entry of $\underline{x}$, in the induced model, makes a $q^{-s(v(a_1)+\cdots+v(a_r))}$ pop out, and this is exactly $|g|^s$, for g in the upper triangular Borel. In other words, appealing to Iwasawa decomposition, $f_{q^{-s}\underline{x}}(g) = |g|^s f_{\underline{x}}(g)$, so that

$$\lambda_{q^{-s}\underline{x}}(\phi) f_{q^{-s}\underline{x}}(g) = \int_G \phi(h) f_{q^{-s}\underline{x}}(gh)\mathrm{d}h = \int_G \phi(h)|gh|^s f_{\underline{x}}(gh) = \lambda_{\underline{x}}(\phi)|g|^s f_{\underline{x}}(g), \tag{18}$$

as claimed. □

Proposition (Central action). *We have, for all $u \in \mathbf{Z}$ and $g \in G$,*

$$\lambda_{\underline{x}}(\phi^u) = (x_1 \cdots x_r)^u \lambda_{\underline{x}}(\phi) \tag{19}$$

where $\phi^u(h) = \phi(\varpi^{-u} g)$.

Proof. In the induced model, we clearly have

$$f_{\underline{x}}\left(\mathrm{diag}(\varpi^u, \ldots, \varpi^u) g\right) = (x_1 \cdots x_r)^u f_{\underline{x}}(g). \tag{20}$$

Hence, we can write, changing variables,

$$\begin{aligned}\int_G f_{\underline{x}}(gh)\phi^u(h)\mathrm{d}h = \int_G f_{\underline{x}}(\omega^u gh)\phi(h)\mathrm{d}h &= (x_1 \cdots x_r)^u \int_G f_{\underline{x}}(gh)\phi(h)\mathrm{d}h \\ &= (x_1 \cdots x_r)^u \lambda_{\underline{x}}(\phi) f_{\underline{x}}(g),\end{aligned}$$

as claimed. □

2.6. **Three properties.** The above Whittaker functions $W_{\underline{x}}(g)$ have three "invariance" properties that will be critical. We state them in this section and provide complete proofs.

Proposition (Inverse-transpose). *Let w be the longest Weyl element of G. We have*

$$W_{\underline{x}}(w^t g^{-1}, \psi) = W_{\underline{x}^{-1}}(g, \bar{\psi}) \tag{21}$$

where $\underline{x}$ denotes the tuple $(x_1^{-1}, \ldots, x_n^{-1})$.

Proof. By uniqueness, we only need to check that the above function on the left of (21) satisfies the defining properties of the Whittaker function on the right. First, by K-invariant and since $w \in K$, it is straightforward that $W(w^t I^{-1}) = W(w) = 1$. Second, using the left-action by N,

$$W_{\underline{x}}(w^t (ng)^{-1}) = W_{\underline{x}}(w^t n^{-1} {}^t g^{-1}) = \theta(w^t n^{-1} w^{-1}) W_{\underline{x}}(w^t g^{-1}). \tag{22}$$

Moreover, $\theta(w^t n^{-1} w^{-1}) = \bar{\theta}(n)$, justifying that we added back ψ in the notation for the statement above, and this proves the first defining property of $W(\cdot, \bar{\psi})$. We remove ψ from here on.

For the third property, we have by change of variables

$$\int_G W_{\underline{x}}(w^t (gh)^{-1})\phi(h)\mathrm{d}h = \int_G W_{\underline{x}}(w^t g^{-1} h)\phi({}^t h^{-1})\mathrm{d}h = \int_G W_{\underline{x}}(w^t g^{-1} h)\check{\phi}(h)\mathrm{d}h, \tag{23}$$

where $\check{\phi}(g) = \phi(w^t g^{-1} w^{-1})$, where the long Weyl elements have been added using the bi-K-invariance of ϕ. Using the defining property of W, we obtain

$$\int_G W_{\underline{x}}(w^t g^{-1} h)\check{\phi}(h)\mathrm{d}h = \lambda_{\underline{x}}(\check{\phi}) W_{\underline{x}}(w^t g^{-1}). \tag{24}$$

Using the property proved in the previous section, we have that $\lambda_{\underline{x}}(\check{\phi}) = \lambda_{\underline{x}^{-1}}(\phi)$, finishing the proof. □

Proposition (Homogeneity). *We have*

$$W_{q^{-s}\underline{x}}(g) = |g|^s W_{\underline{x}}(g). \tag{25}$$

Proof. We use a similar argument, appealing to the uniqueness of $W_{\underline{x}}$. It is clear by definition that the function $|g|^{-s}W_{q^{-s}\underline{x}}(g)$ takes value 1 at $g = I$. The second property, i.e. the left-action by N, is unharmed by the extra factors since $|n| = 1$ for all $n \in N$.

For the third property, we use the proposition of the previous section to write

$$\int_G W_{q^{-s}\underline{x}}(gh)\phi(h)\mathrm{d}h = \lambda_{q^{-s}\underline{x}}(\phi)W_{q^{-s}\underline{x}}(g) = \lambda_{\underline{x}}(\phi)|g|^s W_{\underline{x}}(g) = \int_G |gh|^s W_{\underline{x}}(gh)\phi(h)\mathrm{d}h$$

so that both functions $W_{q^{-s}\underline{x}}(g)$ and $|g|^s W_{\underline{x}}(g)$ are equal. □

Proposition (Central action). *We have*

$$W_{\underline{x}}\left(\mathrm{diag}(\varpi^u, \ldots, \varpi^u)g\right) = (x_1 \cdots x_r)^u W_{\underline{x}}(g). \tag{26}$$

Proof. Since the element is central, it does not change the left-action of N by θ. Use the analogous property for $\lambda_{\underline{x}}$ to write

$$\begin{aligned}
\lambda_x(\phi)W_{\underline{x}}(\mathrm{diag}(\varpi^u, \ldots, \varpi^u)g) &= \int_G W_{\underline{x}}(\mathrm{diag}(\varpi^u, \ldots, \varpi^u)gh)\phi(h)\mathrm{d}h \\
&= \int_G W_{\underline{x}}(gh)\phi^u(h)\mathrm{d}h \\
&= (x_1 \cdots x_r)^u \lambda_{\underline{x}}(\phi)W_{\underline{x}}(g)
\end{aligned}$$

so that we obtain the desired statement. □

2.7. **Translating into polynomials.** An important fact is that, through the Satake isomorphism, many things here can be naturally seen as polynomials in the parameters x_i.

Shintani proves an explicit formula for W that makes it a polynomial in the "variables" (and this is generalized by Miyauchi for unramified, and Matringe with another method, and boils down to the Casselman-Shalika formula).

This setting ought to be clarified.

References

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