LECTURE 2 — BERNSTEIN–ZELEVINSKY CLASSIFICATION

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Abstract. This talk will recall and present some essential results on supercuspidal representations, and will culminate with the classification of irreducible representations of GL_n.

The aim of the talk is threefold:

- introduce supercuspidal representations
- prove that irreducible representations are admissible
- give a recipe producing all the irreducible representations: they are cooked up from supercuspidals (this is the content of the Bernstein–Zelevinsky classification)

1. Sueprcuspidal representations

Let F be a local non-Archimedean field, i.e. a finite extension of $\mathbf{Q}_p$ or of $\mathbf{F}_p((t))$. Let $G = \mathrm{GL}_n(F)$, and $(\pi, V) \in \mathrm{Rep}(G)$ a smooth representation of G. Recall that the contragredient $(\pi^\vee, V^\vee)$ is the set of smooth linear functionals (and not only the dual, i.e. the linear functionals). We will denote generically $v \in V$ and $v^\vee \in V^\vee$.

Definition (Cuspidal)**.** The representation π is called *cuspidal* if for all proper parabolic $P = MU$ of G, the Jacquet functor is $\pi_U := J_P(\pi) = \{0\}$. This is equivalent to

$$\nexists P \subsetneq G, \nexists \tau \in \mathrm{Rep}(M), \pi \hookrightarrow \mathrm{Ind}_P^G(\tau), \tag{1}$$

i.e. a cuspidal representation is a representation that cannot be realized as a subrepresentation of an induced.

Proof. The equivalence between the defintion and the characterization (1) is a consequence of the fact that induction and Jacquet functor are adjoint functors, i.e. Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\tau)) = \mathrm{Hom}_P(\pi, \tau) = \mathrm{Hom}_M(\pi_U, \tau) \tag{2}$$

so that the definition of cuspidal (which means that the term on the right is zero) is equivalent to (1) (which states that the left term is zero). □

Definition (Supercuspidal)**.** An irreducible representation π is called *supercuspidal* if does not occur as a sub-quotient in $\mathrm{Ind}_P^G(\tau)$ for P proper parabolic of G and $\tau \in \mathrm{Rep}(M)$.

Remark. It is a non-trivial fact, consequence of the Bernstein–Zelevinsky classification, that irreducible cuspidal representation are supercuspidal (at least in characteristic 0).

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These notes have been taken on the go by Didier Lesesvre. They may contain typos and errors.

Definition (Compact modulo center)**.** We say that π is *compact* (resp. *compact modulo center*) if for all $v \in V$ and all open compact subgroup K of G,

$$D_{v,K}(g) := \pi(\mathbf{1}_K g^{-1}) \cdot v := \mathrm{vol}(K)^{-1} \int_K \pi(kg^{-1}) v \mathrm{d}k$$

has compact (resp. modulo center) support.

Remark. Note that this is a projection operator on K-fixed vectors.

Definition (Matrix coefficient)**.** Given $v \in V$ and $v^\vee \in V^\vee$, define the associated matrix coefficient

$$m_{v,v^\vee}(g) = \langle \pi^\vee(g) v^\vee, v \rangle = \langle v^\vee, \pi(g^{-1}) v \rangle := (\pi^\vee(g) v^\vee)(v) \in \mathbf{C}, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket.

By smoothness of $v, v^\vee$, the matrix coefficient factorize by some open compact subgroups K' and K of G as follows:

$$\begin{array}{ccc} C & \xrightarrow{\quad m_{v,v^\vee} \quad} & \mathbf{C} \\ & \searrow \qquad \nearrow & \\ & K' \backslash G / K & \end{array}$$

Recall the Cartan decomposition, with $K^\circ = \mathrm{GL}_2(\mathfrak{O})$:

$$G = \bigsqcup_{a_1 \geqslant \cdots \geqslant a_n} K^\circ \begin{pmatrix} \varpi^{a_1} & & & \\ & \varpi^{a_2} & & \\ & & \ddots & \\ & & & \varpi^{a_n} \end{pmatrix} K^\circ \tag{4}$$

Hence the double cosets appearing in the above diagram are parametrized by

$$K' \backslash G / K \simeq \left\{ k_i' \gamma^a k_j \ : \ a \in \mathbb{Z}^{\geqslant}, i \in I, j \in J \right\} \tag{5}$$

where I and J are finite sets, and $\mathbf{Z}^{\geqslant}$ stand for the set of integers tuples satisfying the condition appearing in the above union. We don't really care about this ordering ordering since we can swap the elements in the diagonal matrix by the action of the Weyl group, which is in K°. Typically, for $n = 2$,

$$\mathrm{GL}_2(\mathbf{Z}_p) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \mathrm{GL}_2(\mathbf{Z}_p) = \mathrm{GL}_2(\mathbf{Z}_p) \begin{pmatrix} p & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbf{Z}_p) \tag{6}$$

since the Weyl element $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ is in $\mathrm{GL}_2(\mathbf{Z}_p)$.

Lemma (Characterization by coefficients)**.** A representation π has compact support (mod center) if and only if the matrix coefficients $m_{v,v^\vee}$ have compact support (mod center).

Proof. ($\Longrightarrow$) By smoothness there exists $K \subset G$ such that $v^\vee \in (V^\vee)^K$. Hence in particular $v^\vee = \pi^\vee(\mathbf{1}_K) v^\vee$ so that $m_{v,v^\vee}(g) = \langle v^\vee, D_{v,K}(g) \rangle$.

($\Longleftarrow$) is slightly harder. The aim is to find $(v_i^\vee)_i$ a finite family of vectors so that $\mathrm{supp}(D_{v,K}) \subset \bigcup_i \mathrm{supp}(m_{v_i^\vee, v})$, in particular $\mathrm{supp}(D_{v,K})$ will be compact since each $\mathrm{supp}(m_{v_i^\vee, v})$ is by assumption. It suffices to show that $\mathrm{Im}(D_{v,K})$ is contained in a vector space of finite dimension. By contradiction, assume there is $(g_i)_{i \in \mathbb{N}}$ an infinite family of elements of $K' \backslash G / K$ such that the $D_{v,K}(g_i)$ are linearly independent.

By (5), the sequence $(g_i)_i$ does not belong to any compact, because ϖ^a and its powers belong to a open compact only if $a = 0$. We just need to construct a coefficient which is non-zero on these g_i's. Define a linear form $v^\vee \in (V^\vee)^K$ such that $\langle \pi^\vee, D_{v,K}(g_i)\rangle = 1 = m_{v,v^\vee}(g_i)$ (which is possible since the $D_{v,K}(g_i)$ are linearly independent) and extend it by zero elsewhere on V. Then the matrix coefficient $m_{v,v^\vee}$ does not have compact support, since it contains $\bigcup_i D_{v,K}(g_i)$. This gives a contradiction. □

Corollary. *If π is compact, then* $\mathrm{Im}(D_{v,K})$ *is contained in a finite dimensional vector space.*

Proof. This is a consequence of the previous proof. □

Recall that π is finitely generated means that V is generated by the translates $\pi(g)v_i$ for $g \in G$ and finitely many $v_i \in V$. Irreducibles are in particular generated by any non-zero element. Recall that admissible means that for each open compact subgroup K, we have that V^K is finite dimensional.

Proposition. *π finitely generated and compact, then it is admissible.*

Proof. Assume $V = \langle \pi(g^{-1})v_i\rangle_{g\in G,i}$ is finitely generated. Since $\pi(\mathbf{1}_K)$ projects onto V^K, we have

$$V^K = \pi(\mathbf{1}_K)V = \langle \pi(\mathbf{1}_K g^{-1})v_i\rangle_{g\in G,i} = \langle D_{v_i,K}(g)\rangle_{g\in G,i}$$

which is included in a finite sum of finite dimensional spaces by the above corollary. □

Theorem (Harish-Chandra)**.** *Let $G^\circ = \{g \in G \ : \ \det(g) \in \mathfrak{O}^\times\}$. A representation of G° is compact if and only if it is cuspidal.*

Remark. G° contains all standard unipotents, so cuspidal also makes sense on G°. More generally, $\pi \in \mathrm{Rep}(G)$ is compact modulo the center if and only if it is cuspidal.

Proof. Let's admit it: this is mostly based on the Cartan decomposition. □

Corollary. *Any irreducible cuspidal $\pi \in \mathrm{Irr}(G)$ is admissible.*

Proof. As V is irreducible, thus finitely generated, and the index $[G : ZG^\circ]$ is finite, it follows that V is finitely generated as ZG°-representation as well. But Z acts on V scalarly (by the central character), hence $\pi_{|G^\circ}$ is finitely generated.

By the Harish-Chandra theorem, since π is cuspidal then it is compact. By the proposition, since $\pi_{|G^\circ}$ is compact and finitely generated, thus it is admissible. □

Theorem. *Any irreducible $\pi \in \mathrm{Irr}(G)$ is admissible.*

Proof. Take a minimal parabolic $P = MU$ such that $\pi_U \neq 0$. By the equivalent in the definition of cuspidal), i.e. that $\pi \hookrightarrow \mathrm{Ind}_P^G(\tau)$, there exists $\tau \in \mathrm{Irr}(M)$ which is a quotient of π_U. Note that here P may be equal to the whole G, and it precisely happens in the supercuspidal case. It is now enough to show that this induced $\mathrm{Ind}_P^G(\tau)$ is admissible (as a subrepresentation of an admissible is clearly also admissible).

However, by Iwasawa decomposition, induction preserves admissibility (since they preserve the K-invariants). It is therefore enough to show that $\tau \in \mathrm{Rep}(M)$ is admissible. But by the corollary, since τ is irreducible, it is enough to show that τ is cuspidal.

If τ was not cuspidal, there would exist a smaller standard parabolic $P = M'U' \subsetneq P$ such that $\tau_{U'\cap M} \neq 0$. Note that $U' = U \cdot (U' \cap M)$. By transitivity of the Jacquet functor:

$$\pi_{U'} = (\pi_U)_{U'\cap M} \twoheadrightarrow \tau_{U'\cap M} \neq 0,$$

which would contradict the minimality of $P = MU$ with respect to π. □

2. Bernstein-zelevinsky classification

After having noticed the interest of compactly supported matrix coefficients, we are led to relax a bit this property. Bernstein-Zelevinsky did most of the job, and Langlands extracted and packaged them in a nice way needed for the theory of automorphic representations.

Recall that the *central character* ω of π is defined by the character such that, for all central elements $z \in Z$, for all $v \in V$, we have $\pi(z)v = \omega(z)v$ (for irreducible representations, it exists by Schur's lemma).

Definition (Essentially square integrable and discrete series)**.** A representation having a central character ω is *essentially square integrable* if there is a character $\chi : F^\times \to \mathbf{R}_+^\times$ such that

$$\int_{Z\backslash G} |m_{v,v^\vee}(g)|^2 \chi(\det(g)) \mathrm{d}g < \infty. \tag{7}$$

Moreover, if χ can be taken to be the trivial character $\mathbf{1}$, i.e. if

$$\int_{Z\backslash G} |m_{v,v^\vee}(g)|^2 \mathrm{d}g < \infty, \tag{8}$$

then we call π *square integrable*, or say it is a *discrete series*.

The space of square integrable representations (with character ω) is denoted $L^2(Z\backslash G, \omega)$.

Remark. In fact, χ is unique and the only χ that can work is $\chi^{-1} = |\omega|^2$ by the above formula (this is the only way for the integrand to be Z-invariant). In the square integrable case, the central character has to be unitary.

Proposition. *An irreducible representation π is essentially square integrable if and only if there exists one non-zero coefficient which is L^2.*

Proof. see e.g. Knightly-Li. □

Definition (Segments)**.** A *segment* is $\Delta = (\sigma, \sigma(1), \dots, \sigma(b-1))$ where σ is a supercuspidal representation of $\mathrm{GL}_a(F)$ and $b \in \mathbb{N}^\star$.

We say that the segments Δ and Δ' are *linked* if neither Δ or Δ contains the other, and if $\Delta \cup \Delta'$ is a segment, in particular $a = a'$. In other words, one precedes the other and they overlap nontrivially.

We say that a segment Δ *precedes* Δ' if they are liked and there is $r \geqslant 1$ so that $\sigma' = \sigma(r)$.

Theorem (Bernstein-Zelevinsky)**.** *Let $P = P(n_1, \dots, n_k) = MU$ a parabolic subgroup of G associated with the partition $n = n_1 + \cdots + n_k$. Consider $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ where σ_i is a supercuspidal representation of* $\mathrm{GL}_{n_i}(F)$*.*

1. *The induction $\mathrm{Ind}_P^G(\sigma)$ is reducible if and only if there is $i \neq j$ such that $\sigma_i = \sigma_j(1)$ where we define $\sigma(k) := \sigma \otimes |\det|_F^k$, for all $k \in \mathbf{Z}$.*
2. *The induction of a segment $\mathrm{Ind}_P^G(\Delta)$ is reducible and has a unique irreducible quotient denoted $Q(\Delta)$, and this $Q(\Delta)$ is essentially square integrable. Moreover, every essentially square integrable representation is obtained this way, for a unique triple (a, b, σ).*
3. *Consider segments Δ_i so that Δ_i does not precede Δ_j for $i < j$. Then the induced representation $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$ has a unique irreducible quotient, called the* Langlands quotient *and denoted $Q(\Delta_1, \dots, \Delta_k)$. Any irreducible representation π of G is obtained uiniquely in this way (up to permutation of the Δ_i).*

(4) *The Langlands quotient* $Q(\Delta_1, \dots, \Delta_k)$ *is generic if and only if the induction is irreducible, i.e. it is not the proper quotient, i.e. it is a full induction. This is equivalent to "no two* Δ_i *are linked".*

Example. For instance, with $n = 2$, we have

$$\mathbf{1} \longrightarrow \mathrm{Ind}_B^G \begin{pmatrix} |\cdot|^{-1/2} & \\ & |\cdot|^{1/2} \end{pmatrix} \longrightarrow \mathrm{St}. \tag{9}$$

Remark. Ind_P^G is the I_P^G of Julien's talk (i.e. the normalized induction).

More generally, the Steinberg St_n for GL_n is the representation corresponding to $a = 1$, $\sigma = |\cdot|^{\frac{1-n}{2}}$ (i.e. starting in the middle, so that the segment is unitary) and $b = n$.

Also, if $b = 1$, then $Q(\Delta) = \sigma$ and these are discrete series.

References

[1] Bernstein and Zelevinsky *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I*, Annales scientifiques de l'École Normale Supérieure (1977), 441–472.

[2] Joseph Bernstein *Representations of* $\mathfrak{p}$*-adic groups*, Lectures at Harvard University, 1992.