LECTURE 1 — CASSELMAN AND GL(2)

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ABSTRACT. We discuss the paper of Casselman, proving the existence of newforms for GL(2), the associated multiplicity one theorem and discussing the relation with the classical L-functions and the notion of conductor of a representation. We introdude the necessary background before move towards the main arguments of the proofs.

Let F be a non-Archimedean local field, $\mathfrak{O}$ its ring of integers, ϖ a uniformizer, q the cardinality of the residue field, and the abolute value is normalized so that $|\varpi| = q^{-1}$. Let $G = \mathrm{GL}_2/F$, B its Borel subgroup of upper triangular matrices, T the diagonal torus inside B and U the unipotent radical of unipotent upper triangular matrices, so that $B = T \rtimes U$. Denote Z the center of G, mode of scalar matrices. Hence

$$B = \begin{pmatrix} \star & \star \\ & \star \end{pmatrix} \qquad U = \begin{pmatrix} 1 & \star \\ & 1 \end{pmatrix} \qquad T = \begin{pmatrix} \star & \\ & \star \end{pmatrix} \qquad Z = \begin{pmatrix} x & \\ & x \end{pmatrix}. \tag{1}$$

1. Representations of G

All the representations considered in this talk are smooth.

1.1. Parabolic induction and Jacquet functor.

Let (σ, V) a representation of T. By inflation, it defines a representation of $B = T \rtimes U$, acting trivially on the U-part. By induction, we obtain $\mathrm{Ind}_B^G(\sigma)$ which underlying vector space is the space of locally constant functions $g : G \to V$ such that

$$f(bg) = \sigma(b) f(gh), \qquad b \in B, \ g \in G. \tag{2}$$

The action is then by right translations $g \cdot f : x \mapsto f(xg)$. This is still smooth since the quotient $B \backslash G$ is compact. This is called the *parabolic induction*, and defines a representation of G.

The Jacquet functor goes the other way. Take (π, V) a representation of G. By restriction, we obtain a representation (π_B, V) of B. Looking at the U-coinvariants (largest quotient of V on which U acts trivially), i.e.

$$J(\pi) := V_U := V / \langle u \cdot v - v \rangle_{u \in U, v \in V} \tag{3}$$

gives the *Jacquet functor* of π. Then $(\pi, J(\pi))$ is a smooth T-representation.

Proposition. *We have the following properties:*

- *Both functors preserve admissibility.*
- *Both functors are exact (in particular, the central term of a short exact sequence is admissible if and only if the two side terms are admissible).*

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- *(Frobenius reciprocity) These two functors are adjoint, i.e.*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G(\pi)) \simeq \mathrm{Hom}_T(J(\pi), \sigma). \tag{4}$$

1.2. **Finite dimensional representations of G.** Finite dimensional representations of G are easy to describe.

Proposition. *Any smooth finite dimensional representation of G factors through the determinant, i.e. is of the form $\mu \circ \det$ for μ a character of $F^\times$, in particular it is one-dimensional.*

Proof. A representation of U can be identified with a representation of F, same for $\bar{U}$. And U and $\bar{U}$ generate SL_2. (argument to be written) □

1.3. **The Steinberg representation.** Consider $\mathrm{Ind}_T^G(\mathbf{1}_T)$, i.e. functions $G \to \mathbf{C}$ that are locally constant and left-B-invariant. Inside $\mathrm{Ind}_T^G(\mathbf{1}_T)$ lies the G-invariant line (i.e. a subrepresentation) of constant functions $\mathbf{C} \cdot \mathbf{1}_G$. The Steinberg representation is the remaining quotient:

Definition. We define

$$\mathrm{St} := \mathrm{Ind}_T^G(\mathbf{1}_T)/\mathbf{C} \cdot \mathbf{1}_G. \tag{5}$$

Proposition. St *is an irreducible admissible representation of G.*

1.4. **Principal series.** For χ_1, χ_2 two characters of $F^\times$, define $\chi_1 \otimes \chi_2 : T \to \mathbf{C}^\times$ a smooth character of the torus by

$$\chi_1 \otimes \chi_2 \begin{pmatrix} a & \\ & b \end{pmatrix} = \chi_1(a)\chi_2(b). \tag{6}$$

Define the normalized parabolic induced

$$I(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi_1|\cdot|^{1/2} \otimes \chi_2|\cdot|^{-1/2}). \tag{7}$$

where the normalization is chosen so that such an induction preserves unitarizability.

Proposition. *We have the following properties*

- *If $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, then $I(\chi_1, \chi_2)$ is irreducible. These representations are called the* principal series *of G. We have $I(\chi_1, \chi_2) = I(\psi_1, \psi_2)$ if and only if $(\chi_1, \chi_2) = (\psi_1, \psi_2)$ or (ψ_2, ψ_1), and these are the only equivalences among principal series.*
- *If $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, i.e. $\chi_1|\cdot|^{1/2} = \chi_2|\cdot|^{-1/2} =: \chi$, then there is a nonsplit short exact sequence*

$$0 \longrightarrow \chi \circ \det \longrightarrow I(\chi_1, \chi_2) \longrightarrow \mathrm{St}(\chi \circ \det) \longrightarrow 0. \tag{8}$$

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$$0 \longrightarrow \mathrm{St}(\chi \circ \det) \longrightarrow I(\chi_1, \chi_2) \longrightarrow \chi \circ \det \longrightarrow 0. \tag{9}$$

1.5. **Supercuspidal representations.**

Definition. An irreducible admissible representation π of G is supercuspidal if it is not a subquotient (i.e. a subrepresentation or a quotient) of $I(\chi_1, \chi_2)$ for some characters $\chi_1, \chi_2 : F^\times \to \mathbf{C}^\times$.

Proposition. *If π is an irreducioble admissible representation of G, then it is supercuspidal if and only if $J(\pi) = 0$.*

Proof. This follows from Frobenius reciprocity. □

We deduce from this definition a classification of representations of G:

- one-dimensional representations $\chi \circ \det$
- principal series $I(\chi_1, \chi_2)$
- twists of the Steinberg $\mathrm{St}(\chi \circ \det)$
- supercuspidals

2. Whittaker models

Many of the results recalled here are from Jacquet-Langlands. Let (π, V) be a smooth representation. Fix $\psi : F \simeq U \to \mathbf{C}^\times$ be a nontrivial smooth additive characters.

Definition. A Whittaker functional for (π, V) with respect to ψ is a (generally non-smooth) linear form $\Lambda : V \to \mathbf{C}$ such that

$$\Lambda(\pi(u)v) = \psi(u)\Lambda(v), \qquad u \in U,\ v \in V. \tag{10}$$

In other words, we can inject $\pi \hookrightarrow \mathrm{Ind}_U^G \psi$.

Theorem (Multiplicity one)**.** *If (π, V) is irreducible admissible infinite-dimensional, then the space of Whittaker functional is one-dimensional.*

From now on, we assume (π, V) irreducible admissible infinite-dimensional (thus generic).

Definition (Whittaker model)**.** The representation (π, V) is equivalent to

$$\mathcal{W}(\pi, \psi) := \{W_v : g \in G \mapsto \Lambda(\pi(g)v) \in \mathbf{C} \ : \ v \in V\} \tag{11}$$

acting on it by right translations. This is the *Whittaker model* of π.

Proposition. *If $v \neq 0$, then there exists $a \in F^\times$ such that $W_v \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \neq 0$.*

Proof. See my talk, Fourier expansion. □

There Kirillov model of π is then defined by

$$\mathcal{K}(\pi, \psi) := \{\phi : a \in F^\times \mapsto W_v \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \in \mathbf{C} \ : \ v \in V\}. \tag{12}$$

We can explicitly describe the action of B on this space: for $\phi \in \mathcal{K}(\pi, \psi)$, and denoting ε the central character of π, we have

- $\pi \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \phi(x) = \phi(ax)$, for all $a \in F^\times$,
- $\pi \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \phi(x) = \psi(bx)\phi(x)$, for all $b \in F$,
- $\pi \left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right) \phi(x) = \varepsilon(a)\phi(x)$, for all $a \in F^\times$.

In order to get the full action of G, we also need to determine the action of the long Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ which is computed in Jacquet-Langlands in the case of supercuspidal representations, in terms of the Fourier transform on $C_c^\infty(F^\times)$ (and this is particularly critical, could we give a cleverly readable formula?)

Proposition. *Let $\phi \in \mathcal{K}(\pi, \psi)$. Then we have*

- *ϕ is locally constant.*
- *There is $c > 0$ such that $\phi(y) = 0$ if $|y| > c$.*
- *$\phi \in \mathrm{Ker}(V \to J(V))$ if and only if there is $\varepsilon > 0$ such that $\phi(y) = 0$ if $|y| < \varepsilon$.*

Proof. Recall that $F^\times \simeq \mathbf{Z} \times \mathfrak{O}^\times$, so that compacity implies that the $\mathbf{Z}$-part is bounded by above and below. □

Upshot. If π is supercuspidal, then $\mathcal{K}(\pi,\psi)$ lies into $C_c^\infty(F^\times)$ and it is an equality since the latter space is an irreducible representation of B (thus in particular supercuspidal representations are irreducible also as B-representations).

In general, we have

$$0 \longrightarrow C_c^\infty(F^\times) \longrightarrow \mathcal{K}(\pi,\psi) \longrightarrow J(V) \longrightarrow 0 \tag{13}$$

and $J(V)$ is two-dimensional for principal series, one-dimensional for Steinberg and zero-dimensional for supercuspidal (see Godement's notes on Jacquet-Langlands).

3. The local results of Casselman

Let $K = \mathrm{GL}_2(\mathfrak{O})$. For $n \geqslant 0$, let

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \ : \ c \equiv 0 \mod \varpi^n \right\}. \tag{14}$$

Theorem. *Let (π, V) be an irreducible admissible infinite-dimensional representation of G. Then there exists $n \in \mathbb{N}$ such that*

$$V^n := \left\{ v \in V \ : \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n),\ \pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \varepsilon(d) v \right\} \neq 0. \tag{15}$$

Moreover, if n_0 is the smallest such integer, then

$$\dim V^{n_0+i} = 1 + i = \tau(\varpi^i). \tag{16}$$

3.1. **Sketch of the proof.** Casselman says that the existence of n is obvious, which it is not that immediate. **Question.** Find a clear and explicit argument to explain why this is true, so that it is written somewhere.

For the supercuspidal case, it is pretty explicit and he uses the Kirillov model, fixing $\psi : U \to \mathbf{C}^\times$ of conductor $\mathfrak{O}$. The explicit action on the Kirillov model reads

$$\pi \begin{pmatrix} a & x \\ 0 & d \end{pmatrix} \phi(y) = \varepsilon(b)\psi(ab^{-1}y), \qquad \phi \in C_c^\infty(F^\times), \begin{pmatrix} a & x \\ 0 & d \end{pmatrix} \in B, y \in F^\times. \tag{17}$$

Let $m \geqslant 1$ and introduce

$$H = \begin{pmatrix} & 1 \\ -\varphi^m & \end{pmatrix}. \tag{18}$$

Lemma. Let $\mu_1, \mu_2 : \mathfrak{O}^\times \to \mathbf{C}^\times$ be smooth characters with conductors contained (**Question.** or contains?) in $\mathfrak{p}^m$. Then the following are equivalent:

- $\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \mu_1(a)\mu_2(d) v$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$
- Both conditions
  - $\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} v = \mu_1(a)\mu_2(d) v$ for all $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \cap K$
  - $\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} Hv = \mu_1(d)\mu_2(a) v$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$

We want to apply this lemma with $\mu_1\mu_2 = \varepsilon$.

To be continued