LECTURE 0 — A BRIEF OVERVIEW

DIDIER LESESVRE

ABSTRACT. I briefly introduce the aims and spirit of the theories of newforms and the typical results we will be exploring during the workgroup.

Let G be a reductive group and k a non-archimedean local field. Of critical importance is the existence of distinguished lines inside nice representations of $G(k)$, and this is what the theory of newforms is about.

Let π be a nice (e.g. irreducible, admissible, smooth and generic) representation of G.

Existence of newforms. The fundamental question is if there exists a sequence of groups K_n, decresing in size, so that

$$\pi^{K_n} \neq 0 \tag{1}$$

for n large enough, i.e. so that π admits nonzero fixed vectors by the K_n for n large enough. Letting n_π for the smallest n for which this happens, nonzero vectors fixed by K_{n_π} are called *newforms* (and are related to the classical notion of newforms due to Atkin and Lehner).

The smoothness of the representation ensures that this is the case for some filtration of neighborhoods of the identity, but it is interesting to take K_n as large as possible so that this property holds. The question of what would determine a "good choide" of such K_n's remains mysterious.

Uniqueness of newforms. A typical effect of such a choice would be that $(K_n)_n$ decreases slowly enough so that

$$\dim \pi^{K_{n_\pi}} = 1, \tag{2}$$

the so-called multiplicty one theorem, or unicity of newforms.

Description of oldforms. The natural question arising then is: do we know the multiplicities of *oldforms*, i.e. of fixed vector by smaller such subgroups? In other word, for all n, do we have an explicit formula for

$$\dim \pi^{K_n} \ ? \tag{3}$$

The question of how oldforms are related to newforms are pretty interesting, and it is hoped that newforms generate, in some sense, all the oldforms in an explicit way. This is the case for instance for $\mathrm{GL}(n)$ or $GSp(4)$, where some "level raising operators" span the oldforms when applied to newforms. So this question could be phrased as

Are the oldforms generated by the newforms and some explicit operators?

Date: March 23, 2022.

Notes updated on March 24, 2022.

These notes have been taken on the go by Didier Lesesvre. They may contain typos and errors.

Relation to ε-factors. The newforms have pretty rich properties, and they are expected to carry a good amount of the complexity and information of the representation itself. For instance, n_π is called the conductor and is related to the usual notion of conductor coming from the epsilon factor attached to the associated L-function: both are exactly the same one. In other words, if $\varepsilon(s,\pi)$ denotes the epsilon factor associated with the L-function $L(s,\pi)$ attached to π, then it takes the form

$$\varepsilon(s,\pi) = \varepsilon(\pi) q_F^{-n_\pi(s-1/2)}. \tag{4}$$

Test vectors for Whittaker functions. The newforms have also a property of being good "test vectors" for Whittaker functions, in the sense that if v is a newform in π, then the attached Whittaker function is so that

$$W_v \neq 0. \tag{5}$$

This is particularly important to obtain explicit formulas such as Fourier expansion of Shintani/Miyauchi formulas, that depend on W_v and hence are of better use when they are not trivially zero.

Relation with L-functions. Recall that there is a notion of zeta integrals $Z(s,W)$ built from Whittaker functions attached to π, and from which Godement and Jacquet (and others beyond GL(n)) defined the L-function $L(s,\pi)$ as the common denominator of such zeta integrals, i.e. as the generator of the ideal $\langle Z(s,W)\rangle_W$ of $\mathbf{C}[q^s, q^{-s}]$.

Even though this defines the L-function and allows to access to some of its rich properties, this is far from giving explicit formulas (they are defined as the generator of a given ideal). Strickingly, the (suitably normalized) newvector v does this job:

$$L(s,\pi) = Z(s, W_v). \tag{6}$$

Overview of what has been done so far. These classical questions have been answered most importantly for GL(2) with the seminal paper of Casselman, extended to GL(n) by Jacquet, Piatetski-Shapiro and Shalika.

Some few other cases have been explored and found (sometimes very partial) positive answers of the same kind. These are $SL(2)$, $U(2$, $U(3)$, $GSp(4)$ and (very partially) $SO(2n+1)$.

But much remain to be understood and explored, and this is our aim.