QUADRATIC TWISTS OF CENTRAL VALUES: NUMBER FIELDS CASE

Chan Ieong Kuan and Didier Lesesvre

We prove that a cuspidal automorphic representation of GL(3) over any number field is essentially determined by the quadratic twists of its central value. This generalizes a result of Chinta and Diaconu that was valid only over $\mathbf{Q}$ and for Gelbart-Jacquet lifts.

CONTENTS

1. INTRODUCTION

1.1 STATEMENT OF THE RESULT

Given an automorphic object, its associated L-function is built in a way to be a generating function with good analytic properties [?]. Of considerable importance is its value at the central point [?], that appears to carry a lot of information. For instance, in the automorphic forms setting, the non-vanishing at the central point of the L-function attached to a Rankin-Selberg product $\pi_1 \times \pi_2$ is conjectured to be related to the non-vanishing of automorphic periods of arithmetic significance [?]. With a geometric flavor, given an elliptic curve, the Birch and Swinnerton-Dyer conjecture relates the order of vanishing of its associated L-function at the central point to the rank of the elliptic curve. A last example borrowed from algebraic number theory is that the central value of an L-function attached to a quadratic character is related to important invariants of the underlying quadratic field, by the means of the class number formula.

Following this philosophy, we are naturally interested in the following question: does the central value of the L-function attached to an automorphic form determine it? Obviously the single central value shall not be enough since two automorphic forms can be rescaled in order for their central values to match without them being isomorphic. A finite family of twists of the central value is also an elusive bet, since there are infinitely many linearly independent automorphic forms. The question turns therefore to be: is there an infinite family of twists of the central value that entirely determines the conjugacy class of an automorphic form?

The first result in this direction is due to Luo and Ramakrishnan for holomorphic modular forms [**?**, Theorem A]. Let $S_k(N)$ be the space of holomorphic cusp forms of level N, weight k and trivial central character. They proved that if all the central values of the L-function twisted by these quadratic characters are equal up to a constant, then the automorphic forms are equal.

Theorem 1 (Luo-Ramakrishnan)**.** *Let f and f' be normalized newforms in $S_k(N)$ and $S_{k'}(N')$ respectively. Let $L(s,f)$ and $L(s,f')$ be their associated completed L-functions. If there is a nonzero constant κ such that for every quadratic character χ of conductor prime to NN',*

$$L\left(\frac{1}{2}, f\otimes\chi\right) = \kappa\cdot L\left(\frac{1}{2}, f'\otimes\chi\right), \tag{1.1}$$

then $k=k'$, $N=N'$ and $f=f'$.

Based on the machinery developed by Fisher and Friedberg for function fields [**?**] and adapting the methods developed by Chinta and Diaconu in the automorphic representations setting [**?**], Li [**?**, Theorem 1.1] proved an analogous result for automorphic representations of $\mathrm{GL}(2)$ over general number fields. More precisely, let X be a family of quadratic characters as defined in [**?**, Section 1.2]. Li proved that an automorphic representation of $\mathrm{GL}(2)$ is determined by the associated quadratic twists of its central value.

Theorem 2 (Li)**.** *Let π and π' be two self-contragredient cuspidal automorphic representations of $\mathrm{GL}(2)$ over a number field F with trivial central character. Let $L(s,\pi)$ and $L(s,\pi')$ be their associated completed L-functions. Assume there is a character $\chi\in X$ such that the ε-factor $\varepsilon(\frac{1}{2},\pi\otimes\chi)$ is nonzero. If there exists a nonzero constant κ such that for every $\chi\in X$,*

$$L\left(\frac{1}{2}, \pi\otimes\chi\right) = \kappa\cdot L\left(\frac{1}{2}, \pi'\otimes\chi\right), \tag{1.2}$$

then $\pi\simeq\pi'$.

In a similar fashion, Chinta and Diaconu [**?**, Theorem 1.1] proved that a cuspidal automorphic representation of $\mathrm{GL}(3)$ over $\mathbf{Q}$ that is a Gelbart-Jacquet lift is also determined by sufficiently many quadratic twists of the central value of its L-function. More precisely, let X be the set of quadratic Dirichlet characters.

Theorem 3 (Chinta-Diaconu). *Let π and π' be two self-contragredient cuspidal automorphic representations of* $\mathrm{GL}(3)$ *over* $\mathbf{Q}$ *with trivial central character. Let $L(s,\pi)$ and $L(s,\pi')$ be their associated completed L-functions. Assume that π is a Gelbart-Jacquet lift. If there exists a nonzero constant κ such that for every $\chi \in X$,*

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) = \kappa \cdot L\left(\frac{1}{2}, \pi' \otimes \chi\right), \tag{1.3}$$

then $\pi \simeq \pi'$.

Remarks. Put aside the need of the extra non-vanishing assumptions, we can make the following remarks on the article of Chinta and Diaconnu :

(i) This result still holds if we consider partial L-functions instead of completed automorphic L-functions. It is also possible to restrict the family of characters to almost all of them, asking for their conductors to be prime to a fixed number. Moreover, Li proves a slightly stronger result, considering also the case of a general central character. Since we are interested here in the possibility of generalizing Chinta and Diaconu's result to general number fields, already requiring extensive details, we do not add these technical variations.

(ii) One of the aims of the paper of Chinta and Diaconu was to explain the critical difficulties they faced when trying to extend their methods to more general number fields. The proof heavily relies on the machinery of multiple Dirichlet series as developed by Bump, Diaconu, Friedberg, Goldfeld and Hoffstein [**?**, **?**]. However, the lack of good enough bounds of type Lindelöf-on-average for the mean square of central values was a critical obstacle in order to go beyond the case of the number field $\mathbf{Q}$. Building on a quadratic large sieve inequality on number fields due to Goldmakher and Louvel [**?**], generalizing a result of Heath-Brown [**?**], we are able to overcome these difficulties. This is also the opportunity to fill in some gaps usually left under cover of the standardness of the method and that we believe to be written nowhere. We now state the main result of this paper. Let X be the set quadratic Hecke characters as defined in Section 2.1.

Theorem 4. *Let π and π' be two self-contragredient cuspidal automorphic representations of* $\mathrm{GL}(3)$ *over a number field F with trivial central character. Let $L(s,\pi)$ and $L(s,\pi')$ be their associated completed L-functions. Let S be a set containing the ramification places of π. Assume that there is a nonzero constant κ*

such that for every $\chi \in X$,

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) = \kappa \cdot L\left(\frac{1}{2}, \pi' \otimes \chi\right), \tag{1.4}$$

then

- *if* π *is a Gelbart-Jacquet lift, then* $\pi \simeq \pi'$ *;*
- *if* π *is not a Gelbart-Jacquet lift, and there is an ideal ray class* E *such that* $L(1/2, \pi \otimes \chi_E) \neq 0$*, then* $\pi \simeq \pi'$.

Remarks. This result, in particular the restriction to Gelbart-Jacquet lifts, appeal some comments.

(i) Recall that an automorphic representation π of $\mathrm{GL}(3)$ with trivial central character is a Gelbart-Jacquet lift [**?**] of a cuspidal automorphic representation of $\mathrm{GL}(2)$ if and only if its symmetric square L-function $L(s, \pi, \mathrm{sym}^2)$ has a pole at 1, by a result of [**?**].

(ii) Munshi and Sengupta [**?**] recently proved that a $\mathrm{GL}(3)$ Hecke-Maass form is entirely determined by the twists of its central values by Dirichlet characters. However, they result is not only reduced to the base field $\mathbf{Q}$ but also requires to twist by all the primitive characters, not only quadratic ones.

(iii) The question of the size of the family of twists of the central value necessary to determine an automorphic representation is interesting. The results of Luo and Ramakrishnan and the one of Li state that a $\mathrm{GL}(2)$ representation can be determined by quadratic twists. The family of automorphic representations of $\mathrm{GL}(3)$ is larger, but the one made of Gelbart-Jacquet lifts considered here can be seen as a family of $\mathrm{GL}(2)$ objects. To give an idea of the strength of this requirement, recall from [**?**] that the family of all automorphic representations of $\mathrm{GL}(3)$ of bounded spectral parameter T is of asymptotic size of order T^5, while the one made of all the Gelbart-Jacquet lifts if of size T^4. Therefore, it could be expected that a larger family of twists would be necessary to determine a $\mathrm{GL}(3)$ automorphic form without restriction. However, the lack of reciprocity laws for the coefficients of automorphic forms of higher ranks does not allow do to so with the methods used in this article, see Section 3.

(iv) One could wonder whether it is possible to aim at a similar result for cuspidal automorphic representations of higher ranks $\mathrm{GL}(n)$ for $n \geqslant 4$. The proof is based on the meromorphic continuation of the double Dirichlet series attached to the quadratic twists of the central values, and is essentially reduced to a double computation of the residue at the point $(\frac{1}{2}, 1)$. A general fact in the theory of multiple Dirichlet series is that there is not enough known functional equations for $\mathrm{GL}(n)$, with $n \geqslant 4$, to get a meromorphic continuation around this point, so that the current state of the art in this theory does not allow to address the problem for higher ranks with the same methods. See [?] for more details on this spectral barrier problem.

1.2 NON-VANISHING CONSEQUENCE

There is no hope that quadratic twists of the central value can determine an automorphic representation, even in the case of $\mathrm{GL}(2)$: there are cuspidal automorphic representations π on $\mathrm{GL}(2)$ with $L(1/2,\pi\otimes\chi)=0$ for every quadratic character χ. This is the case for instance if π if the base change lift to an imaginary quadratic field of the representation associated to an holomorphic cusp form on $\mathrm{SL}(2,\mathbf{Z})$ as a consequence of the work of [?], see for instance [?]. To avoid this possibility, a non-vanishing assumption is therefore necessary and is actually already present in the $\mathrm{GL}(2)$ case in [?]. Indeed, he assumes that there is a quadratic character such that $\varepsilon(1/2,\pi\otimes\chi)$ is one. A result of [?] ensures that this assumption implies that there are infinitely many nonzero twists $L(1/2,\pi\otimes\chi)$ of the central value.

Non-vanishing questions are often difficult. [?] proved that an L-function $L(s,\pi)$ attached to an automorphic L-function on $\mathrm{GL}(n)$ ouver $\mathbf{Q}$ has infinitely many non-zero twists of its central value $L(1/2,\pi\otimes\chi)$ where χ is a Dirichlet character. However, nothing is said about the order of the character and showing non-vanishing results for quadratic characters remained out of reach. Similarly as the $\mathrm{GL}(2)$ case, it is illusory to expect a non-vanishing result for quadratic twists without further assumptions. As a consequence of our methods to treat the non-Gelbart-Jacquet lift case, we prove the following result in Section 7.4.

Theorem 5. *Let π be a cuspidal automorphic representation of $\mathrm{GL}(3)$ over F. Assume that π is not a Gelbart-Jacquet lift. Assume that there is a class E such that $L(1/2,\pi\otimes\chi_E)\neq 0$. Then there are infinitely many classes $D\in I(S)$ such that $L(1/2,\pi\otimes\chi_D)\neq 0$.*

1.3 OUTLOOK OF THE PROOF

The necessary knowledge on the quadratic characters χ_D for general number fields, parametrized by integer ideals D, and the background on automorphic L-functions are summarized in Section 2. The structure of the proof of Theorem 4 relies on Chinta and Diaconu's original paper [?] concerned with the field of rational numbers, and on the theory of multiple Dirichlet series as developped in [?]. The double Dirichlet series associated to the L-functions $L(s,\pi\otimes\chi_D)$, for a quadratic character χ_D, is defined by

$$\widetilde{Z}(s,w;\pi)=\sum_D\frac{L(s,\pi\otimes\chi_D)}{|D|^w}=\sum_{D,N}\frac{a_\pi(N)\chi_D(N)}{|N|^s|D|^w}, \tag{1.5}$$

where the sums run over nonzero integer ideals of F, the norm for ideals of F is denoted $|\cdot|$, and the $a_\pi(N)$ are the Fourier coefficients of π. These double Dirichlet series are introduced in Section 3.

The GL(3) L-function $L(s,\pi\otimes\chi_D)$ satisfies a functional equation $s\to 1-s$, so that the double Dirichlet series $\widetilde{Z}(s,w;\pi)$ is expected to satisfy an analogous functional equation. On the other hand, the Dirichlet series made of $\mathrm{GL}(1)$ L-function obtained by formal application of the quadratic reciprocity law, namely

$$\widehat{Z}(s,w;\pi)=\sum_{D,N}\frac{a_\pi(N)\chi_N(D)}{|N|^s|D|^w}=\sum_N\frac{L(w,\chi_N)a_\pi(N)}{|N|^s}, \tag{1.6}$$

is closely related to the original sum $\widetilde{Z}(s,w;\pi)$. The $\mathrm{GL}(1)$ L-function $L(w,\chi_n)$ satisfies a functional equation $w\to 1-w$, so that $\widehat{Z}(s,w;\pi)$, and therefore $\widetilde{Z}(s,w;\pi)$, is expected to satisfy an analogous functional equation as well. However this is not the case, since functional equations only make the squarefree part of the ideals D appear. It is therefore necessary to complete the double Dirichlet series, introducing correcting factors $a(s,D,\pi)$ and $b(w,N,\pi)$ given by Dirichlet polynomials and such that the corrected double Dirichlet series takes the form

$$Z(s,w;\pi)=\sum_D\frac{L(s,\pi\otimes\chi_D)}{|D|^w}a(s,D,\pi)=\sum_N\frac{a_\pi(N)L(w,\chi_N)}{|N|^s}b(w,N,\pi), \tag{1.7}$$

and satisfies the expected functional equations. These requirements yield necessary conditions on the desired correcting factors, and Bump, Friedberg and Hoffstein [**?**] proved that there are unique such Dirichlet polynomials $a(s,D,\pi)$ and $b(w,N,\pi)$ in the case of $\mathrm{GL}(3)$. The two expected functional equations are therefore proven for the completed double Dirichlet series $Z(s,w;\pi)$ in Section 4. They relate

$$Z(s,w;\pi)\quad\text{and}\quad Z(\phi(s,w);\pi),$$
$$Z(s,w;\pi)\quad\text{and}\quad Z(\psi(s,w);\pi),$$

where

$$\phi(s,w)=\left(1-s,w+3s-\frac{3}{2}\right),$$
$$\psi(s,w)=\left(s+w-\frac{1}{2},1-w\right).$$

Thanks to these functional equations, it is possible to meromorphically continue the corrected double Dirichlet series $Z(s,w;\pi)$ to the whole complex plane, and this is the purpose of Section 5, using the symmetries brought by the functional equations in order to extend the original domain of convergence. However, there is no obvious path to pull back analytic properties of this corrected double Dirichlet series $Z(s,w;\pi)$ to the original double Dirichlet series $\widetilde{Z}(s,w;\pi)$. A technical sieving process, carried on in Section 6, allows to do so by expressing $\widetilde{Z}(s,w;\pi)$ as a twisted sum of functions of the form $Z(s,w;\pi)$. However, this sieving expresses $\widetilde{Z}(s,w;\pi)$ as an infinite sum of functions of type $Z(s,w;\pi)$, and the convergence of this sum could be not strong enough to ensure the expected meromorphic continuation, so that precise estimates in the different aspects appearing in the summation are necessary: these are precisely established in Section 6.2. This is enough to show that $\widetilde{Z}(s,w;\pi)$ admits a meromorphic continuation around the point $(\frac{1}{2},1)$. It is in this process that the difficulties faced by Chinta and Diaconu have been overcome.

The proof then reduces to a computation of the residues $\widetilde{Z}(s,w;\pi)$ at $(\frac{1}{2},1)$ carried on in Section 7. Indeed, after twisting π by the character χ_N for a certain ideal N, the residue is proven to be an injective function of the N-th Fourier coefficient $a_\pi(N)$ of π for large enough N. Therefore, if we assume that for every character $\chi_D \in X$,

$$L\left(\frac{1}{2},\pi\otimes\chi_D\right) = \kappa\cdot L\left(\frac{1}{2},\pi'\otimes\chi_D\right), \tag{1.8}$$

for a certain nonzero constant κ, then summing over D yields an equality between the associated double Dirichlet series $\widetilde{Z}(\frac{1}{2},w,\pi)$ and $\widetilde{Z}(\frac{1}{2},w,\pi')$, and therefore between their residues. In particular we can deduce the equality of almost all their Fourier coefficients by the injectivity mentioned above, and conclude by multiplicity one.

1.4 ACKNOWLEDGEMENTS

We thank Farrell Brumley and Adrian Diaconu for enlightening discussions. The authors are grateful to the Sun Yat-Sen University for its warm environment.

2. BACKGROUND

2.1 QUADRATIC SYMBOLS ON NUMBER FIELDS

Over $\mathbf{Q}$, the primitive quadratic characters [?] are given by the Legendre symbol $\chi_d(n) = \left(\frac{d}{n}\right)$ for d varying among quadratic discriminants. Given a number field F, these characters have been generalized to the quadratic symbol of the form $\chi_d(N) = \left(\frac{d}{N}\right)$ when $d \in F$ and N is a fractional ideal of F out of ramified primes, see for instance [?]. It is necessary for our purposes to extend the definition of these quadratic characters to symbols of the form $\chi_D(N) = \left(\frac{D}{N}\right)$ for both D and N integer ideals of F, without assuming D principal anymore. This construction has been carried on by Fischer and Friedberg [**?**, Section 1] for function fields, and adapted to number fields by Chinta, Friedberg and Hoffstein [**?**, Section 2]. We provide here an account of their construction, adding details that we believe useful and not available explicitly in the existing literature.

Lemma 1. *There exists a finite set S of finite places, such that $\mathscr{O}^S$ has class number one.*

Proof. Let h be the class number of $\mathscr{O}$. If I is a non-principal ideal of $\mathscr{O}$, then I^h is principal, so that $I^h = (a)$ for an $a \in F$. Let $S = \{P_1, \dots, P_k\}$ be the finite set of prime ideals containing (a), therefore containing I. In the Dedekind domain $\mathscr{O}^S$, made of integer ideals of F prime to S, the ideal (a), and therefore also I, becomes trivial. The class group of $\mathscr{O}$ surjects onto the one of $\mathscr{O}^S$, and the surjection kills I, one of the nontrivial elements of the class group. Thus we get a Dedekind domain $\mathscr{O}^S$ with a smaller class number than $\mathscr{O}$. We finish by induction, adding at each step a finite number of primes to the set S, until the class number reaches one. □

Let S be a finite set of places, containing the archimedean ones and the even ones. By Lemma 1, it can be chosen so that $\mathscr{O}^{S_f}$ has class number one, where S_f denotes the set of finite places in S. For an element a of F, the Kronecker symbol $\left(\frac{a}{\cdot}\right)$ is well-defined [**?**] as the totally multiplicative function trivial on units and satisfying, for P a prime ideal of F,

$$\left(\frac{a}{P}\right) = a^{\frac{|P|-1}{2}} \mod P. \tag{2.1}$$

It remains to extend it as a function of ideals in the numerator. For a place v of F, let F_v be the local completion at v. If v is a finite place, let P_v denote the maximal ideal of F_v and q_v be the cardinality of its residue field. Let $C \in \mathscr{O}$ be defined by

$$C = \prod_{v \in S_f} P_v^{n_v}, \tag{2.2}$$

where the exponents n_v are chosen so that

$$\begin{cases} n_v = 1 & \text{if } v \text{ is not lying above 2;} \\ n_v & \text{such that } \operatorname{ord}_v(a-1) \geqslant n_v \Longrightarrow a \in F_v^{\times 2}. \end{cases} \tag{2.3}$$

Let $I(S)$ be the group of fractional ideals of F prime to S. Let P_C the subgroup of $I(S)$ consisting of totally positive principal ideals (a) with $a \equiv 1$ modulo C. Let $R_C = I(S)/P_C$ be the ray class group modulo C, which is known to have finite cardinality. Let $H_C = R_C \otimes (\mathbf{Z}/2\mathbf{Z}) \simeq I(S)/I(S)^2 P_C$, so that $\widehat{H}_C$ consists of the quadratic characters of R_C. Since H_C is a finite abelian group, it decomposes as a direct sum of cyclic ones: there are elements $\bar{E}_1, \dots, \bar{E}_k$ in H_C such that

$$H_C = \langle \overline{E}_1 \rangle \oplus \cdots \oplus \langle \overline{E}_k \rangle. \tag{2.4}$$

Let $\mathscr{E}_0 = \{E_1, \dots, E_k\}$ be a set of representatives for these generators, with $E_j \in I(S)$. Since $\mathscr{O}^{S_f}$ has class number one, *i.e.* is a principal domain, for every $E_0 \in \mathscr{E}_0$ there is a generator $m_{E_0} \in \mathscr{O}^{S_f}$ such that $E_0\mathscr{O}^{S_f} = m_{E_0}\mathscr{O}^{S_f}$. Let $\mathscr{E}$ be a full set of representatives for R_C, namely the monoid generated by $\mathscr{E}_0$. More precisely, $\mathscr{E}$ is composed of the elements of the form

$$E = \prod_{E_0 \in \mathscr{E}_0} E_0^{n_{E_0}} \quad \text{with} \quad n_{E_0} \geqslant 0. \tag{2.5}$$

For such an element E, define $m_E = \prod_{\mathscr{E}_0} m_{E_0}^{n_{E_0}}$, so that for every $E \in \mathscr{E}$, we get $E\mathscr{O}^{S_f} = m_E\mathscr{O}^{S_f}$. We can now extend the definition of the quadratic symbol. Any ideal D in $I(S)$ decomposes as $(m)EG^2$ with $m \in F^\times$ such that $m \equiv 1 \mod C$ (corresponding to the component in P_C), $E \in \mathscr{E}$ a representative in $I(S)/I(S)^2P_C$, and $G \in I(S)$ being the squareful part. We can now define the quadratic symbol appealing to the already known definition of $\left(\frac{\cdot}{D'}\right)$ for elements of F and $D' \in I(S)$ by letting, for all ideal $D \in I(S)$ decomposed as above,

$$\chi_D(D') = \left(\frac{D}{D'}\right) = \left(\frac{m \cdot m_E}{D'}\right). \tag{2.6}$$

All the good properties expected from a quadratic character are satisfied, as stated in the following proposition, coming from the very definition (2.6) and the properties of the classical quadratic symbol. It is analogous to the corresponding property of Fisher and Friedberg [**?**, Lemma 1.1] for function fields.

Proposition 1. *For D and D' integer ideals in I_C, we have the properties:*

(i) *χ is multiplicative: for every ideals D and D', we have $\chi_{DD'} = \chi_D\chi_{D'}$;*
(ii) *χ only depends on squarefree parts: if D has squarefree part D_0, $\chi_D = \chi_{D_0}$;*
(iii) *reciprocity law: for every D and D' relatively prime ideals, $\chi_D(D')\chi_{D'}(D)$ takes value ± 1 and depends only on the classes of D and D' in H_C.*

The quadratic reciprocity law can be expressed as follows. Define, for two ideals D and D' relatively prime,

$$\eta(D, D') = \chi_D(D')\chi_{D'}(D), \tag{2.7}$$

then $\eta(D, D') = \pm 1$ depends only on the classes of D and D' in H_C. In particular we get

$$\chi_D(D') = \eta(D, D')\chi_{D'}(D). \tag{2.8}$$

2.2 L-FUNCTIONS

2.2.1 • GL(1) CASE

For D a squarefree ideal of $I(S)$, let χ_D be the primitive ray class quadratic character over F as defined in Section 2.1. For every prime ideal P and $w \in \mathbf{C}$, define the local L-factor by

$$L_P(w, \chi_D) = \left(1 - \frac{\chi_D(P)}{|P|^w}\right)^{-1}. \tag{2.9}$$

Let $L_f(w, \chi_D)$ be the finite part of the associated L-function, defined by

$$L_f(w, \chi_D) = \prod_P L_P(w, \chi_D) = \sum_N \frac{\chi_D(N)}{|N|^w}. \tag{2.10}$$

where the product is over prime ideals P of $\mathscr{O}$ and the sum over non zero ideals N of $\mathscr{O}$. It converges on a right half-plane by the convexity bounds. The completed L-function

$$L(w, \chi_D) = \left(\frac{2^{r_1}|D_F|}{(2\pi)^d}\right)^{w/2} \Gamma\left(\frac{w}{2}\right)^{r_1} \Gamma(w)^{r_2} L_f(w, \chi_D), \tag{2.11}$$

is known [?] to have meromorphic continuation to the whole complex plane, more precisely a continuation to an analytic function on $\mathbf{C}\setminus\{1\}$ and a simple pole at 1 if and only if χ_D has conductor one. Moreover the completed L-function satisfies the functional equation

$$L(w, \chi_D) = \varepsilon(w, \chi_D) L(1 - w, \chi_D), \tag{2.12}$$

where the ε-factor is defined by [?, Equation (3.1)]

$$\varepsilon(w, \chi_D) = |D_0|^{\frac{1}{2} - w}, \tag{2.13}$$

and ε_D is a complex number of modulus one. For all finite set S of places, introduce the prime-to-S partial L-function

$$L^S(w, \chi_D) = \prod_{P \in I(S)} L_P(w, \chi_D) = \sum_{N \in I(S)} \frac{\chi_D(N)}{|N|^w}. \tag{2.14}$$

The functional equation (2.12) translates to the partial L-function $L^S(w, \chi_D)$ into

$$L^S(w, \chi_D) = \varepsilon(w, \chi_D) L^S(1 - w, \chi_D) \prod_{v \in S} \frac{L_v(1 - w, \chi_D)}{L_v(w, \chi_D)}, \tag{2.15}$$

and this last product is a meromorphic function.

2.2.2 • GL(3) CASE

Let π be a self-contragredient cuspidal automorphic representation of $\mathrm{GL}(3)$ over F. Gelfand, Graev and Pyatetskii-Shapiro [?] introduced the finite-part L-function attached to π. For every prime ideal P, introduce the local L-factor

$$L_P(s,\pi)=\prod_{j=1}^{3}\left(1-\frac{\gamma_j(P)}{|P|^s}\right)^{-1}, \tag{2.16}$$

where the $\gamma_j(P)$ are the local spectral parameters of π, also called its Satake parameters when π_P is unramified. The best known bound towards the Ramanujan conjecture for $\mathrm{GL}(3)$ over general number fields is due to Blomer and Brumley [?, Theorem 1] and is given by

$$\gamma_j(P)\ll |P|^{5/14+\varepsilon}. \tag{2.17}$$

Introduce for later purposes the convenient notation, for $M\in I(S)$ and $j\in\{1,2,3\}$,

$$\gamma_j(M)=\prod_{P^r||M}\gamma_j\left(P^r\right). \tag{2.18}$$

Moreover, we make the convention that if the indices j appear without being explicitly defined in a sum (resp. product), then it is understood that the expression is a sum (resp. product) over $j\in\{1,2,3\}$. The finite-part L-function attached to π is defined [?] by the Euler product of local L-factors, namely

$$L_f(s,\pi)=\prod_P L_P(s,\pi)=\sum_N\frac{a_\pi(N)}{|N|^s}, \tag{2.19}$$

where the sum runs over nonzero integer ideals of F and the product over prime ideals P. The $a_\pi(N)$ are called the Fourier coefficients of π. For D an integer ideal, the quadratic twist by χ_D of its L-function is defined by

$$L_f(s,\pi\otimes\chi_D)=\sum_N\frac{a_\pi(N)\chi_D(N)}{|N|^s}=\prod_P\prod_{j=1}^{3}\left(1-\frac{\gamma_j(P)\chi_D(P)}{|P|^s}\right)^{-1}. \tag{2.20}$$

Let $c(\pi)$ the arithmetic conductor of π, and let ε_π be the root number of π which is a complex number of modulus one depending only on π. The L-function above admits a completion $L(s,\pi\otimes\chi_D)$, adding archimedean factors $L_v(s,\pi\otimes\chi_D)$ for $v|\infty$, made of explicit Euler gamma functions. This completed L-functions is entire for cuspidal automorphic representations π and satisfies the functional equation [?, Equation (3.3)]

$$L(s,\pi\otimes\chi_D)=\varepsilon(s,\pi\otimes\chi_D)L(1-s,\pi\otimes\chi_D), \tag{2.21}$$

where the ε-factor is defined by

$$\varepsilon(s,\pi\otimes\chi_D)=\varepsilon_\pi|D_0|^{3\left(\frac{1}{2}-s\right)}c(\pi)^{\frac{1}{2}-s}. \tag{2.22}$$

For a finite set S of places, this translates to a functional equation for the partial L-function, given by

$$L^S(s,\pi\otimes\chi_D)= \varepsilon(s,\pi\otimes\chi_D)L^S(1-s,\pi\otimes\chi_D)\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_D)}{L_v(s,\pi\otimes\chi_D)}, \tag{2.23}$$

where the last product is a meromorphic function.

3. DOUBLE DIRICHLET SERIES

3.1 PURE AND CORRECTED DOUBLE DIRICHLET SERIES

Let π be an automorphic cuspidal representation of $\mathrm{GL}(3)$ over F. Let S be a finite set of places as chosen in Section 2.1, so that the quadratic symbols χ_D are defined for $D\in I(S)$. From now on, all the ideals considered in this article are in $I(S)$, in particular all the summation variables appearing are assumed to be in $I(S)$, unless otherwise stated. Define the double Dirichlet series associated to the partial L-function $L^S(s,\pi\otimes\chi_D)$

$$\widetilde{Z}^S(s,w;\pi)=\sum_{D\in I(S)}\frac{L^S(s,\pi\otimes\chi_D)}{|D|^w}. \tag{3.1}$$

that converges for $\sigma=\Re(s)>1$ and $\tau=\Re(w)>1$. All the double Dirichlet series appearing in this article are convergent on this domain, and s and w lie in it unless otherwise stated. Writing the L-function attached to $\pi\otimes\chi_D$ as a Dirichlet series (2.20) yields the more explicit expression

$$\widetilde{Z}^S(s,w;\pi)=\sum_{D,M\in I(S)}\frac{a_\pi(M)\chi_D(M)}{|D|^w|M|^s}. \tag{3.2}$$

Following [?, Theorem 2.1], there are unique Dirichlet polynomials $a^S(s,D,\pi,\alpha)$ and $b^S(w,M,\pi,\beta)$ such that, for all character $\alpha,\beta \in \widehat{H}_C$,

$$\sum_{D\in I(S)} \frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w} a^S(s,D,\pi,\alpha)\beta(D)$$

$$= \sum_{M\in I(S)} \frac{L^S(w,\chi_M\beta)}{|M|^s} b^S(w,M,\pi,\beta)\alpha(M), \tag{3.3}$$

$$a^S(s,D,\pi,\alpha) = |D_1|^{3(1-2s)}\chi_\pi(D_1^2)a^S(1-s,D,\pi,\alpha), \tag{3.4}$$

$$b^S(w,M,\pi,\beta) = |M_1|^{1-2s}\chi_\pi(M_1^2)b^S(1-w,M,\pi,\beta). \tag{3.5}$$

In particular, the relation (3.3) is the "basic identity" of Bump, Friedberg and Hoffstein. Moreover, from the explicit description [?, Equations (2.1)-(2.3) pages 29-30 and 37] of these correcting factors, we have the bounds for $\sigma,\nu > 1$,

$$a^S(s,D,\pi,\alpha) \ll_\varepsilon 1, \tag{3.6}$$

$$a^S(1/2,D,\pi,\alpha) \ll_\varepsilon |D_1|^{5/7+\varepsilon}, \tag{3.7}$$

$$b^S(w,M,\pi,\beta) \ll_\varepsilon |M_1|^\varepsilon. \tag{3.8}$$

Introduce therefore the twisted corrected double Dirichlet series, for α and β characters on ideals prime to S,

$$Z^S(s,w;\pi,\alpha,\beta) = \sum_{D\in I(S)} \frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w} a^S(s,D,\pi,\alpha)\beta(D) \tag{3.9}$$

$$= \sum_{M\in I(S)} \frac{L^S(w,\chi_M\beta)}{|M|^s} b^S(w,N,\pi,\beta)\alpha(M). \tag{3.10}$$

Remark. The correction factor $a^S(s,D,\pi,\alpha)$ is trivial for squarefree ideals D, so that this corrected double Dirichlet series amounts to mollify the terms for non-squarefree ideals in (3.1). The double Dirichlet series $Z^S(s,w;\pi,\alpha,\beta)$ and $\widetilde{Z}^S(s,w;\pi)$ are therefore expected to be closely related: this is indeed the case, as shown in Section 6.

3.2 RAY CLASS SELECTION

Let E be a representative of a ray class in H_C, and δ_E the characteristic function of this class, in the sense that for every ideal D, $\delta_E(D)$ takes value 1 if D is in the same class than E modulo $I(S)^2P_C$, and 0 otherwise. The partition of H_C in classes yields

$$1 = \sum_{[E]\in H_C} \delta_E, \tag{3.11}$$

where the summation runs over a set of representatives of classes in H_C. Moreover, letting $h_C = |H_C|$, the orthogonality relations for characters in H_C can be rephrased as, for every element E in H_C,

$$\delta_E = h_C^{-1} \sum_{\rho \in \widehat{H}_C} \rho(E)^{-1} \rho. \tag{3.12}$$

We extend the definition of the double Dirichlet series as well as of the correction factors in order to allow α and β to be such selection functions of characters, even though they are not multiplicative. For any functions δ, δ' on H_C written as sums of characters

$$\begin{aligned} \delta &= \sum_{\rho \in \widehat{H}_C} \lambda_\rho \cdot \rho, \\ \delta' &= \sum_{\rho' \in \widehat{H}_C} \mu_{\rho'} \cdot \rho', \end{aligned}$$

define

$$\begin{aligned} Z^S(s,w;\pi,\alpha\delta,\beta\delta') &= \sum_{\rho,\rho' \in \widehat{H}_C} \lambda_\rho \mu_{\rho'} \cdot Z^S(s,w;\pi,\alpha\rho,\beta\rho') \\ a^S(s,D,\pi,\beta\delta') &= \sum_{\rho' \in \widehat{H}_C} \mu_{\rho'} \cdot a^S(s,D,\pi,\beta\rho') \\ b^S(w,N,\pi,\alpha\delta) &= \sum_{\rho \in \widehat{H}_C} \lambda_\rho \cdot b^S(w,N,\pi,\beta\rho) \end{aligned}$$

Lemma 2 (Sieving by classes in H_C)**.** *Let E, E' be ideals of $I(S)$. We have*

$$Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}) = h_C^{-2} \sum_{\rho,\rho' \in \widehat{H}_C} \rho(E)^{-1} \rho'(E')^{-1} \cdot Z^S(s,w;\pi,\alpha\rho,\beta\rho'),$$

$$Z^S(s,w;\pi,\alpha,\beta) = \sum_{[E],[E'] \in H_C} Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}),$$

where this last sum is over a set of representatives in H_C.

Proof. For the first relation, use the orthogonality relation (3.12) to get

$$\begin{aligned} Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}) &= \sum_{D\in[E']} \frac{L^S(s,\pi\otimes\chi_D\alpha\delta_E)}{|D|^w} a^S(s,D,\pi,\alpha\delta_E)\beta(D) \\ &= h_C^{-1}\sum_D \frac{L^S(s,\pi\otimes\chi_D\alpha\delta_E)}{|D|^w} a^S(s,D,\pi,\alpha\delta_E)\beta(D) \sum_{\rho'\in\widehat{H}_C} \rho'(E')^{-1}\rho'(D) \\ = \quad & h_C^{-1} \sum_{\rho'\in\widehat{H}_C} \rho'(E')^{-1}\sum_D \frac{L^S(s,\pi\otimes\chi_D\alpha\delta_E)}{|D|^w} a^S(s,D,\pi,\alpha\delta_E)\beta(D)\rho'(D). \end{aligned}$$

Now, use the basic identity (3.3) in order to rephrase this last sum into

$$\begin{aligned} & h_C^{-1} \sum_{\rho'\in\widehat{H}_C} \rho'(E')^{-1}\sum_D \frac{L^S(s,\pi\otimes\chi_D\alpha\delta_E)}{|D|^w} a^S(s,D,\pi,\alpha\delta_E)\beta(D)\rho'(D) \\ &\quad = h_C^{-1} \sum_{\rho'\in\widehat{H}_C} \rho'(E')^{-1}\sum_M \frac{L^S(w,\chi_M\rho'\beta)}{|M|^s} b^S(w,M,\pi,\rho'\beta)\alpha(M)\delta_E(M). \end{aligned}$$

Appeal again to the orthogonality relations (3.12) to select ideals in $[E]$ and get

$$\begin{aligned} & h_C^{-1} \sum_{\rho'\in\widehat{H}_C} \rho'(E')^{-1}\sum_M \frac{L^S(w,\chi_M\rho'\beta)}{|M|^s} b^S(w,M,\pi,\rho'\beta)\alpha(M)\delta_E(M) \\ &\quad = h_C^{-2} \sum_{\rho,\rho'\in\widehat{H}_C} \rho(E)^{-1}\rho'(E')^{-1}\sum_M \frac{L^S(w,\chi_M\rho'\beta)}{|M|^s} b^S(w,M,\pi,\rho'\beta)\alpha(M)\rho(M), \end{aligned}$$

and we recognize this last sum as being the claimed double Dirichlet series. The second relation is obvious by partitioning $I(S)$ by classes in H_C. □

This sieving lemma amounts to say that we can afford to select a single class for any variable appearing in the double Dirichlet series at the expense of a finite linear combination. A critical fact is that the linear combination appearing in the above lemma has length and coefficients uniformly bounded in π, D, M, α and β, and therefore enjoy the same convergence properties than the full double Dirichlet series. In this case, we speak of a *uniformly finite linear combination*.

4. FUNCTIONAL EQUATIONS

4.1 EPSILON-FACTORS

Let π be a self-contragredient cuspidal automorphic representation of $\mathrm{GL}(3,\mathbf{A})$ with central character χ_π, that is unramified out of S and moreover assumed to be a principal series representation at every place out of S. It is possible to suppose so without loss of generality [?], since this is the case except for a finite number of places that can be added to S.

Lemma 3 (Class expression for ε-factors)**.** *Let D,E be integer ideals prime to S. Suppose that D and E lie in the same class in H_C, that is to say we have $\chi_D = \chi_E\chi_m$ for a certain $m \in F^\times$ satisfying $m \equiv 1 \pmod{C}$. Let π be a self-contragredient cuspidal automorphic representation of* $\mathrm{GL}(n,\mathbf{A})$ *for $n = 1$ or $n = 3$, unramified outside S. Then $\varepsilon_v(s,\pi_v \otimes \chi_{D,v})$ does not depend on D but only on E for places $v \in S$. Moreover,*

$$\varepsilon(s,\pi\otimes\chi_D) = \chi_\pi\left(\frac{D_0}{E_0}\right)\left|\frac{D_0}{E_0}\right|^{n(\frac{1}{2}-s)} \varepsilon(s,\pi\otimes\chi_E). \tag{4.1}$$

Proof. Note that the result is obvious if the central values are zero, so that we can assume they are nonzero until the end of the proof. By the definition (2.22) of the ε-factor,

$$\varepsilon(s,\pi\otimes\chi_D) = |c(\pi\otimes\chi_D)|^{\frac{1}{2}-s}\,\varepsilon\left(\frac{1}{2},\pi\otimes\chi_D\right). \tag{4.2}$$

Let $v \in S$. Recall that, thanks to the decomposition $D = (m)EG^2$ obtained in Section 2.1, we have $\chi_D = \chi_E\chi_m$. It is assumed that $v_P(m-1) > 0$ for finite places $P \in S$, that is to say $m \equiv 1$ modulo C. Since C is the product of finite primes in S we deduce that, for $v \in S$, we have $\chi_{D,v} = \chi_{E,v}$.

Let $v \notin S$. In that case we make use of the assumption that π_v is an unramified principal series, namely $\pi = \psi_1$ if $n = 1$ and $\pi = \pi(\psi_1,\psi_2,\psi_3)$ if $n = 3$, for characters ψ_j of $F_v^\times$. The ε-factors (2.13) decompose as a product of local ε-factors of the form, for any $j \in \{1,2,3\}$,

$$\varepsilon_v\left(\frac{1}{2},\psi_j\chi_{D,v},\psi_v\right) = \psi_j(P_v)^{\mathrm{ord}_v(c(\chi_D))}\varepsilon_v\left(\frac{1}{2},\chi_{D,v},\psi_v\right). \tag{4.3}$$

The same formula holds for the specific choice $D = E$, so that in particular we get by dividing these nonzero quantities and taking product of j, and since $\psi_1\psi_2\psi_3 = \chi_\pi$,

$$\frac{\varepsilon\left(\frac{1}{2},\pi\otimes\chi_D\right)}{\varepsilon\left(\frac{1}{2},\pi\otimes\chi_E\right)} = \chi_\pi\left(\frac{D_0}{E_0}\right)\frac{\varepsilon\left(\frac{1}{2},\chi_D\right)^n}{\varepsilon\left(\frac{1}{2},\chi_E\right)^n}. \tag{4.4}$$

The central values for quadratic characters are assumed to be nonzero, so that they are equal to one. Moreover,

$$\frac{c(\pi\otimes\chi_D)}{c(\pi\otimes\chi_E)} = \left|\frac{D_0}{E_0}\right|^n, \tag{4.5}$$

yielding the result. ∎

This lemma exactly makes explicit the dependence on a fixed representative in H_C. This remark is a motivation to reduce the study to double Dirichlet series with sums restricted to a single class in H_C, at the cost of uniformly finite linear combinations by Lemma 2. We immediately deduce from this and the fact that $(\pi\otimes\chi_M)\otimes\chi_D = \pi\otimes\chi_{DM}$ the following corollary, which is [?, Corollary 1.11].

Corollary 1. *Let $L\in I(S)$ squarefree. Suppose $D,E\in I(S)$ are in the same class of H_C. Let π be a self-contragredient cuspidal automorphic representation of* $\mathrm{GL}(n,\mathbf{A})$ *for $n=1$ or $n=3$, unramified outside S. Then*

$$\varepsilon(s,(\pi\otimes\chi_L)\otimes\chi_D) = \chi_\pi\left(\frac{D_0}{E_0}\right)\left|\frac{D_0}{E_0}\right|^{n\left(\frac{1}{2}-s\right)}\varepsilon(s,(\pi\otimes\chi_L)\otimes\chi_E). \tag{4.6}$$

4.2 FUNCTIONAL EQUATION IN s

The L-function $L(s,\pi)$ satisfies a functional equation (2.21) relating $s\leftrightarrow 1-s$, so that the associated double Dirichlet series is expected to follow a similar functional equation. Recall that

$$\phi(s,w) = \left(1-s, w+3s-\frac{3}{2}\right). \tag{4.7}$$

Proposition 2 (Functional equation $\phi: s\leftrightarrow 1-s$)**.** *For an automorphic cuspidal representation π on* $\mathrm{GL}(3)$, *α,β characters on ideals, E a class in H_C prime to S, the double Dirichlet series $Z^S(s,w;\pi,\alpha,\beta\delta_E)$ satisfies the functional equation for $\sigma,\tau>1$,*

$$\begin{aligned} Z^S(s,w;\pi,\alpha,\beta\delta_E) &= Z^S(\phi(s,w);\pi,\alpha,\beta\delta_E)\\ &\qquad\times\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3\left(\frac{1}{2}-s\right)}}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}. \end{aligned} \tag{4.8}$$

Proof. This is essentially summing over D the functional equations of the involved L-functions, ε-factors and correcting factors. We follow the steps of Fisher and Friedberg [?, Theorem 3.1]. By (3.4), the epsilon factors satisfy the functional equation

$$a^S(s,D,\pi,\alpha) = a^S(1-s,D,\pi,\alpha)\chi_\pi(D_1^2)|D_1|^{6\left(\frac{1}{2}-s\right)}. \tag{4.9}$$

By Lemma 3, the ε-factors at places in S only depend on the class E of D, not on D itself, so that $L_v(s,\pi\otimes\chi_D) = L_v(s,\pi\otimes\chi_E)$ for every place $v\in S$. We can make explicit the dependences in these places, writing

$$\begin{aligned} & L^S(s,\pi\otimes\chi_D\alpha) && (4.10)\\ &\qquad = L^S(1-s,\pi\otimes\chi_D\alpha)\varepsilon(s,\pi\otimes\chi_D\alpha)\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_D\alpha)}{L_v(s,\pi\otimes\chi_D\alpha)} && (4.11)\\ &\qquad = L^S(1-s,\pi\otimes\chi_D\alpha)\varepsilon(s,\pi\otimes\chi_D\alpha)\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}. && (4.12)\end{aligned}$$

We therefore get, plugging both functional equations together,

$$\begin{aligned} L^S(s,\pi\otimes\chi_D)a^S(s,D,\pi,\alpha) &= L^S(1-s,\pi\otimes\chi_D\alpha)a^S(1-s,D,\pi,\alpha)\\ &\quad\times|D_1|^{6(\frac12-s)}\chi_\pi(D_1^2)\varepsilon(s,\pi\otimes\chi_D\alpha)\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}. \end{aligned} \tag{4.13}$$

Adding also the expression (4.1) of the ε-factor in terms of the representative E in H_C, the factor coming out from the functional equation (4.9) for $a^S(s,D,\pi,\alpha)$ consists in the squareful part of D, and the one coming out from the ε-factor according to Lemma 3 and Corollary 1 consists in the squarefree part of D to the same exponent. So that we get in (4.13),

$$\begin{aligned} L^S(s,\pi\otimes\chi_D\alpha)a^S(s,D,\pi,\alpha) &= L^S(1-s,\pi\otimes\chi_D\alpha)a^S(1-s,D,\pi,\alpha)\\ &\quad\times\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)|E_0|^{3(\frac12-s)}}\chi_\pi(D)|D|^{3(\frac12-s)}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}. \end{aligned} \tag{4.14}$$

In order to recover the double Dirichlet series $Z^S(s,w;\pi,\alpha,\beta\delta_E)$, we need to sum over D. Because of the δ_E is present in the double Dirichlet series, all the summands D are in the same H_C-class than E, in particular for all D in this class, $\chi_D=\chi_E$. After multiplying by $\beta(D)|D|^{-w}$ and summing (4.14) over $D\in I(S)$, for w with large enough real part, we get

$$\begin{aligned} Z^S(s,w;\pi,\alpha,\beta\delta_E) &= \sum_{D\in[E]}\frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w}a^S(s,D,\pi,\alpha)\beta(D)\\ &= \frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{|E_0|^{3(\frac12-s)}}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}\\ &\quad\times\sum_{D\in[E]}\frac{L^S(1-s,\pi\otimes\chi_D\alpha)}{|D|^{w+3s-3/2}}a^S(1-s,D,\pi,\alpha)\beta(D). \end{aligned}$$

We recognize the sought double Dirichlet series $Z^S(\phi(s,w);\pi,\alpha,\beta\delta_E)$ and the claimed functional equation. □

Summing over the classes E in H_C therefore yields an explicit functional equation relating $Z^S(s,w;\pi,\alpha,\beta)$ and $Z^S(\phi(s,w);\pi,\alpha,\beta)$. Since these sums are uniformly finite by Lemma 2, the analytic behavior is similar and we can afford to switch freely between statements on $Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$ and statements on $Z^S(s,w;\pi,\alpha,\beta)$.

It is sometimes necessary to make explicit the dependence on a specific ideal r, for instance this is critical in Proposition 9. When explicit bounds in this specific aspect are needed, we can appeal to the following refined version of the functional equation.

Proposition 3 (Refined functional equation $\phi: s \leftrightarrow 1-s$)**.** *Consider an automorphic cuspidal representation π on $\mathrm{GL}(3)$, α,β characters on ideals, E a class in H_C out of S, and an ideal $r\in I(S)$. Let S_r be the finite set of places in S along with those dividing r, and $f_r(\alpha)$ be the product of primes dividing r and the conductor of α. We have that*

$$\prod_{P|r/f_r(\alpha)}\prod_j\left(1-\frac{\alpha\gamma_j(P)^2}{|P|^{2-2s}}\right)Z^{S_r}(s,w;\pi,\alpha,\beta\delta_E) \tag{4.15}$$

is a uniformly finite linear combination of expressions of the type, for $\rho,\rho'\in\widehat{H}_C$,

$$\sum_{l_j|r/f_r(\alpha)}\frac{\gamma_j(l_j)}{|l_j|^{1-s}}\sum_{m_j|r/f_r(\alpha)}\frac{\mu\gamma_j(m_j)}{|m_j|^s}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}$$
$$\times\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}Z^{S_r}(\phi(s,w);\pi,\alpha\rho,\beta\rho'\chi_\pi\chi_{l_1l_2l_3m_1m_2m_3}).$$

Proof. The refinement compared to the functional equation obtained in Proposition 2 consists in undisclosing the L-factors for the places P dividing r, according to (2.23). Note that, since χ_D is a quadratic character, we have

$$1-\frac{\alpha\gamma_j(P)^2}{|P|^{2-2s}}=\left(1-\frac{\alpha\gamma_j\chi_D(P)}{|P|^{1-s}}\right)\left(1+\frac{\alpha\gamma_j\chi_D(P)}{|P|^{1-s}}\right). \tag{4.16}$$

Moreover, the product of these expressions over $P|r$ can be reduced to a product over $P|r/f_r(\alpha)$ since, for $P|f_r(\alpha)$, the character $\alpha(P)$ vanishes so that the corresponding factor is trivial. Multiplying both sides of the functional equation (4.8) by $\prod(1-\alpha\gamma_j(P)^2|P|^{-2+2s})$ for $P|r/f_r(\alpha)$ and expanding therefore yields

$$\begin{aligned}
&\prod_{P|r/f_r(\alpha)}\left(1-\frac{\alpha\gamma_j(P)^2}{|P|^{2-2s}}\right)Z^{S_r}(s,w;\pi,\alpha,\beta\delta_E)\\
&=Z^{S_r}(\phi(s,w);\pi,\alpha,\beta\delta_E\chi_\pi)\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}\\
&\quad\times\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}\prod_{P|r/f_r(\alpha)}\prod_j\left(1-\frac{\alpha\gamma_j\chi_D(P)}{|P|^s}\right)\left(1+\frac{\alpha\gamma_j\chi_D(P)}{|P|^{1-s}}\right)\\
&=Z^{S_r}(\phi(s,w);\pi,\alpha,\beta\delta_E\chi_\pi)\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}\\
&\quad\times\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}\sum_{l_j|r/f_r(\alpha)}\frac{\alpha\chi_D\gamma_j(l_j)}{|l_j|^{1-s}}\sum_{m_j|r/f_r(\alpha)}\frac{\mu\alpha\chi_D\gamma_j(m_j)}{|m_j|^s}.
\end{aligned}$$

After switching summations and opening up the definition of the double Dirichlet series involved, we get

$$\begin{aligned}
&\prod_{P|r/f_r(\alpha)}\left(1-\frac{\alpha\gamma_j(P)^2\chi_D(P)}{|P|^{2-2s}}\right)Z^{S_r}(s,w;\pi,\alpha,\beta\delta_E)\\
&=\sum_{l_j|r/f_r(\alpha)}\frac{\alpha\chi_D\gamma_j(l_j)}{|l_j|^{1-s}}\sum_{m_j|r/f_r(\alpha)}\frac{\mu\alpha\chi_D\gamma_j(m_j)}{|m_j|^s}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}\\
&\quad\times Z^{S_r}(1-s,w+3s-3/2;\pi,\alpha,\beta\delta_E\chi_\pi)\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}\\
&=\sum_{l_j|r/f_r(\alpha)}\frac{\alpha\gamma_j(l_j)}{|l_j|^{1-s}}\sum_{m_j|r/f_r(\alpha)}\frac{\mu\alpha\gamma_j(m_j)}{|m_j|^s}\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E\alpha)}{L_v(s,\pi\otimes\chi_E\alpha)}\frac{\varepsilon(s,\pi\otimes\chi_E\alpha)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}\\
&\quad\times\sum_D\frac{L^{S_r}(1-s,\pi\otimes\chi_D\alpha)}{|D|^{w+3s-3/2}}a^S(1-s,D,\pi,\alpha)\beta\delta_E\chi_\pi(D)\chi_D(l_1l_2l_3m_1m_2m_3).
\end{aligned}$$

It remains to integrate the quadratic symbol $\chi_D(l_1l_2l_3m_1m_2m_3)$ in the sum over D, seeing it as a character in D in order to recognize a genuine double Dirichlet series. The quadratic reciprocity law for quadratic characters (2.7) writes, for D in the same class than E,

$$\chi_D(l_1l_2l_3m_1m_2m_3)=\eta(E,l_1l_2l_3m_1m_2m_3)\chi_{l_1l_2l_3m_1m_2m_3}(D).\tag{4.17}$$

Thus, up to sieving the classes of $l_1,\cdots,m_3$ in H_C by the orthogonality relations (3.12), that is to say up to a uniformly finite linear combination by Lemma 2, we are reduced to sums of the form

$$\sum_{l_j|r/f_r(\alpha)} \frac{\alpha\gamma_j(l_j)}{|l_j|^{1-s}} \sum_{m_j|r/f_r(\alpha)} \frac{\mu\alpha\gamma_j(m_j)}{|m_j|^s} \prod_{v\in S} \frac{L_v(1-s,\pi\otimes\chi_E)}{L_v(s,\pi\otimes\chi_E)} \frac{\varepsilon(s,\pi\otimes\chi_E)}{\chi_\pi(E_0)^{-1}|E_0|^{3(\frac{1}{2}-s)}}$$
$$\times Z^{S_r}\left(1-s, w+3s-\frac{3}{2}; \pi, \alpha\rho, \beta\rho'\chi_\pi\chi_{l_1l_2l_3m_1m_2m_3}\right),$$

for $\rho,\rho' \in \widehat{H_C}$. This is exactly the claimed result. □

4.3 FUNCTIONAL EQUATION IN w

The L-function $L(w,\chi_D)$ satisfies a functional equation (2.15) relating $w \leftrightarrow 1-w$, so that the same is expected for the associated double Dirichlet series. This is indeed the case by the following proposition. Recall that

$$\psi(s,w) = \left(s+w-\frac{1}{2}, 1-w\right). \tag{4.18}$$

Proposition 4 (Functional equation $\psi : w \leftrightarrow 1-w$)**.** *Consider an automorphic cuspidal representation π on* GL(3)*, α,β characters on ideals, E a class in H_C out of S, and an ideal $r \in I(S)$. Let S_r be the finite set of places in S along with those dividing r, and $f_r(\beta)$ be the product of primes dividing r and the conductor of β. The double Dirichlet series $Z^S(s,w;\pi,\alpha\delta_E,\beta)$ satisfies the functional equation*

$$Z^S(s,w;\pi,\alpha\delta_E,\beta) = Z^S(\psi(s,w);\pi,\alpha\delta_E,\beta) \times \frac{\varepsilon(w,\chi_E)}{|E_0|^{1-2s}} \prod_{v\in S} \frac{L_v(1-w,\chi_E\beta)}{L_v(w,\chi_E\beta)}. \tag{4.19}$$

Proof. The proof is analogous to the one of Proposition 2 and amounts to sum the functional equations for the L-functions $L(w,\chi_M\beta)$ and those for the correcting factors $b^S(w,M,\beta,\pi)$. By (3.3),

$$Z^S(s,w;\pi,\alpha\delta_E,\beta) = \sum_{M\in[E]} \frac{L^S(w,\chi_M\beta)}{|M|^s} \cdot b^S(w,M,\pi,\beta)\alpha(M). \tag{4.20}$$

Introducing the functional equations (2.12) for the L-function $L(w,\chi_M)$ and (3.5) for $b^S(w,M,\pi,\beta)$, and then using Lemma 3 to formulate the ε-factor in function of the representative E, we get

$$\begin{aligned}
&L^S(w,\chi_M\beta)b^S(w,M,\pi,\beta)\\
&=L^S(1-w,\chi_M\beta)b^S(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\\
&\qquad\times\varepsilon(w,\chi_M\beta)|M_1|^{1-2s}\chi_M\beta(M_1^2)\\
&=L^S(1-w,\chi_M\beta)b^S(1-w,M,\pi,\beta)\varepsilon(w,\chi_E\beta)\\
&\qquad\times\chi_M\beta\left(\frac{M_0}{E_0}\right)\left|\frac{M_0}{E_0}\right|^{\frac{1}{2}-s}|M_1|^{1-2s}\chi_M\beta(M_1^2)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}.
\end{aligned}$$

Multiplying by $\alpha(M)|M|^{-w}$ and summing over M yields

$$\begin{aligned}
Z^S(s,w;\pi,\alpha\delta_E,\beta)=\sum_{M\in[E]}\frac{L^S(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}b^S(1-w,M,\pi,\beta)\prod_{P|S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\\
\times\frac{\varepsilon(w,\chi_E\beta)}{|E_0|^{1-2s}\chi_M\beta(E_0)}\chi_M\beta(M)\alpha(M).
\end{aligned}$$

Therefore we recognize the double Dirichlet series

$$\begin{aligned}
Z^S(s,w;\pi,\alpha\delta_E,\beta)=Z^S\left(s+w-\frac{1}{2},1-w;\pi,\alpha\delta_E,\beta\right)\\
\times\frac{\varepsilon(w,\chi_E\beta)}{|E_0|^{1-2s}\beta(E_0)}\prod_{v\in S}\frac{L_v(1-w,\chi_E\beta)}{L_v(w,\chi_E\beta)}.
\end{aligned}$$

which is exactly the claimed result. □

Analogously to Proposition 3, there is a refined version of this functional equation in order to have an explicit dependence in a fixed ideal $r\in I(S)$.

Proposition 5 (Refined functional equation $\psi: w\leftrightarrow 1-w$)**.** *Consider an automorphic cuspidal representation π on* GL(3)*, α,β characters on ideals, E a class in H_C out of S, and an ideal $r\in I(S)$. Let S_r be the finite set of places in S along with those dividing r, and $f_r(\beta)$ be the product of primes dividing r and the conductor of β. We have that*

$$\prod_{P|r/f_r(\beta)}\left(1-\frac{\beta(P)^2}{|P|^{2-2w}}\right)Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta)\tag{4.21}$$

can be written as a uniformly finite linear combination of expressions of the form

$$\sum_{l|r/f_r(\beta)} \frac{\mu(l)\beta(l)}{|l|^w} \sum_{m|r/f_r(\beta)} \frac{\mu(m)^2\beta(m)}{|m|^{1-w}} \times Z^{S_r}(\psi(s,w);\pi,\alpha\delta_E\chi_\pi\chi_{lm},\beta).$$

Proof. The proof is analogous to the one of Proposition 3, and essentially amounts to carefully sum the associated functional equations (4.8) over all M and to undisclose the local L-factors at places $P|r$ according to (2.15). We have

$$Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta) = \sum_{M\in[E]} \frac{L^{S_r}(w,\chi_M\beta)}{|M|^s} b^S(w,M,\pi,\beta)\alpha(M). \tag{4.22}$$

The functional equation (2.12) gives

$$L^{S_r}(w,\chi_M\beta) = \varepsilon(w,\chi_M\beta)L^{S_r}(1-w,\chi_M\beta)\prod_{P|r}\frac{1-\chi_M\beta(P)|P|^{-w}}{1-\chi_M\beta(P)|P|^{-1+w}}\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}.$$

Recall that by (3.5) the local factors $b^{S_r}(w,N,\pi,\beta)$ satisfy the function equation

$$b^{S_r}(w,M,\pi,\beta) = |M_1|^{1-2w}\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta). \tag{4.23}$$

Introducing these two functional equations in the expression (4.22) leads to

$$Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta) = \sum_{M\in[E]} \frac{L^{S_r}(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)} \times\prod_{P|r}\left(1-\frac{\chi_M\beta(P)}{|P|^w}\right)\left(1-\frac{\chi_M\beta(P)}{|P|^{1-w}}\right)^{-1}\alpha(M).$$

Multiplying both sides by $\prod(1-\beta(P)^2|P|^{-2+2w})$, where the product is over primes P dividing $r/f_r(\beta)$, gives

$$\prod_{P|r/f_r(\beta)}\left(1-\frac{\beta(P)^2}{|P|^{2-2w}}\right)Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta)$$
$$= \sum_{M\in[E]} \frac{L^{S_r}(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\alpha(M)$$
$$\times\prod_{P|r/f_r(\beta)}\left(1-\frac{\chi_M\beta(P)}{|P|^w}\right)\left(1-\frac{\chi_M\beta(P)}{|P|^{1-w}}\right)^{-1}\left(1-\frac{\beta(P)^2}{|P|^{2-2w}}\right).$$

Note that the product over $P|r$ can be reduced to a product over $P|r/f_r(\beta)$ since, for $P|f_r(\beta)$, the character $\beta(P)$ vanishes so that the corresponding factor in the product is trivial. Moreover, since the character χ_M is quadratic we have, for primes $P|r/f_r(\beta)$,

$$\left((1-\frac{\beta(P)^2}{|P|^{2-2w}}\right)\left(1-\frac{\chi_M\beta(P)}{|P|^{1-w}}\right)^{-1}=1+\frac{\chi_M\beta(P)}{|P|^{1-w}}. \tag{4.24}$$

Including this in the expression above, expanding the finite Euler product and switching the involved summations yields

$$\begin{aligned}
&\prod_{P|r/f_r(\beta)}\left(1-\frac{\beta(P)^2}{|P|^{2-2w}}\right)Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta)\\
&=\sum_{M\in[E]}\frac{L^{S_r}(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\\
&\quad\times\prod_{P|r}\left(1-\frac{\chi_M\beta(P)}{|P|^{w}}\right)\left(1+\frac{\chi_M\beta(P)}{|P|^{1-w}}\right)\alpha(M)\\
&=\sum_{M\in[E]}\frac{L^{S_r}(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\\
&\quad\times\sum_{l|r/f_r(\beta)}\frac{\mu(l)\chi_M(l)\beta(l)}{|l|^{w}}\sum_{m|r/f_r(\beta)}\frac{\mu(m)^2\chi_M(m)\beta(m)}{|m|^{1-w}}\alpha(M)\\
&=\sum_{l|r/f_r(\beta)}\frac{\mu(l)\beta(l)}{|l|^{w}}\sum_{m|r/f_r(\beta)}\frac{\mu(m)^2\beta(m)}{|m|^{1-w}}\sum_{M\in[E]}\frac{L^{S_r}(1-w,\chi_M\beta)}{|M|^{s+w-\frac{1}{2}}}\\
&\quad\times\chi_\pi(M_1^2)b^{S_r}(1-w,M,\pi,\beta)\prod_{v\in S}\frac{L_v(1-w,\chi_M\beta)}{L_v(w,\chi_M\beta)}\chi_M(lm)\alpha(M).
\end{aligned}$$

It remains to integrate $\chi_M(lm)$ as a character in M, so that we can recognize a genuine double Dirichlet series. In order to do so, we can sum over classes of lm in H_C, which is done at a cost of a uniformly finite linear combination by Lemma 2. We are therefore reduced to sums of the type

$$\sum_{l|r/f_r(\beta)}\frac{\mu(l)\beta(l)}{|l|^{w}}\sum_{m|r/f_r(\beta)}\frac{\mu(m)^2\beta(m)}{|m|^{1-w}}
Z^{S_r}\left(s+w-\frac{1}{2},1-w;\pi,\alpha\delta_E\chi_\pi\chi_{lm},\beta\right),$$

giving the claimed functional equation in w. □

After taking the sum over the ray classes E, we get an explicit functional equation relating $Z^{S_r}(s,w;\pi,\alpha,\beta)$ and $Z^{S_r}(\psi(s,w);\pi,\beta)$. Finally, we got the relation between the double Dirichlet series at (s,w) under the two transformations

$$\phi(s,w) = \left(1-s, w+3s-\frac{3}{2}\right),$$
$$\psi(s,w) = \left(s+w-\frac{1}{2}, 1-w\right).$$

In particular we can verify that the two involutions ϕ and ψ generate a group of functional equations isomorphic to the dihedral group D_6. Indeed, it admits the presentation $\phi^2 = \psi^2 = 1$ and $(\phi\psi)^6 = 1$.

5. MEROMORPHIC CONTINUATION

We now prove that, based on the functional equations satisfied by the corrected double Dirichlet series, established in the previous section, we can extend $Z^S(s,w;\pi,\alpha,\beta)$ meromorphically to the whole $\mathbf{C}^2$. We will repeatedly use the following continuation principle due to Hartog [**?**].

Proposition 6 (Hartog's continuation principle)**.** *Let R be a connected tube domain, that is to say a connected domain of the form $S(\omega) = \{s \in \mathbf{C}^2 : \Re(s) \in \omega\}$ where ω is an open set of $\mathbf{R}^2$. Then any holomorphic function on $S(\omega)$ can be analytically continued to its convex hull $S(\widehat{\omega})$.*

Proposition 7 (Rough meromorphic continuation)**.** *Let α, β be characters of finite order, and E, E' two classes in H_C. The function*

$$(w-1)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}) \tag{5.1}$$

has an analytic continuation to the region R_1 made of the $(s,w) \in \mathbf{C}^2$ in

$$\left\{\begin{array}{lll} \tau > \frac{5}{2} - 3\sigma & \text{if} & \sigma \leqslant -\varepsilon \\ \tau > \frac{5}{2} - \frac{3}{2}\sigma & \text{if} & -\varepsilon < \sigma < 1+\varepsilon \\ \tau > 1 & \text{if} & \sigma \geqslant 1+\varepsilon \end{array}\right\} \cup \left\{\begin{array}{lll} \sigma > \frac{3}{2} - \nu & \text{if} & \tau \leqslant -\varepsilon \\ \sigma > \frac{3}{2} - \frac{1}{2}\nu & \text{if} & -\varepsilon < \tau < 1+\varepsilon \\ \sigma > 1 & \text{if} & \tau \geqslant 1+\varepsilon \end{array}\right\}.$$

Proof. First of all, let us deal with the s-aspect. By definition, we have

$$Z^S(s,w;\pi,\alpha,\beta\delta_{E'}) = \sum_{D\in[E']} \frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w}\beta(D)a^S(s,D,\pi,\alpha). \tag{5.2}$$

On $\sigma > 1+\varepsilon$, the L-functions $L(s,\pi\otimes\chi_D\alpha)$ are uniformly convergent and therefore uniformly bounded. Moreover, β happens to be a finite order character, so that it has norm less than one. Finally, the correcting factor satisfies $a^S(s,D,\pi,\alpha) \ll |D|^{\varepsilon}$ for $\sigma > 1+\varepsilon$ by (3.6). So that the double Dirichlet series $Z^S(s,w;\pi,\alpha,\beta\delta_{E'})$ converges on $\tau > 1+\varepsilon$, and is therefore holomorphic on the region $\sigma > 1$ and $\tau > 1$.

Appealing to the functional equation (4.8), since all the extra factors appearing are holomorphic, we deduce that $Z^S(s,w;\pi,\alpha,\beta)$ is also holomorphic on $\sigma < 0$ and $\tau + 3\sigma > 5/2$. In the remaining region $0 \leqslant \sigma \leqslant 1$, the convexity bound $L(s,\pi\otimes\chi_D\alpha) \ll |D|^{3/2-\sigma}$ for $\sigma < 0$ and the Phragmén-Lindelöf principle yield that $L^S(s,\pi\otimes\chi_D\alpha)$ is bounded by $|D|^{3(1-\sigma)/2}$. In particular, for $\tau > 5/2-3\sigma/2+\varepsilon$, the series $Z^S(s,w;\pi,\alpha,\beta\delta_E)$ uniformly converges and is therefore holomorphic in this region.

This proves that $Z^S(s,w;\pi,\alpha,\beta\delta_{E'})$ is holomorphic in the region $R_{1,1}$ made of all the (s,w) in $\mathbf{C}^2$ such that

$$\begin{cases} \tau > \dfrac{5}{2} - 3\sigma & \text{if} \quad \sigma \leqslant -\varepsilon, \\ \tau > \dfrac{5}{2} - \dfrac{3}{2}\sigma & \text{if} \quad -\varepsilon < \sigma < 1+\varepsilon, \\ \quad \tau > 1 & \text{if} \quad \sigma \geqslant 1+\varepsilon. \end{cases} \tag{5.3}$$

Now, we turn to the formulation of the double Dirichlet series $Z^S(s,w;\pi,\alpha,\beta\delta_{E'})$ in terms of GL(1) L-functions given by (3.10), that is

$$Z^S(s,w;\pi,\alpha\delta_E,\beta) = \sum_{M\in[E]} \frac{L^S(w,\chi_M\beta)}{|M|^s} b^S(w,N,\pi,\beta)\alpha(M). \tag{5.4}$$

The completing factor satisfies $b^S(w,N,\pi,\beta) \ll |N|^{\varepsilon}$ for $\nu > 1+\varepsilon$ by (3.8) and α is of norm bounded by one since it is a finite order character. In the domain $\tau > 1+\varepsilon$, the L-function $L^S(w,\chi_M\beta)$ is uniformly convergent and therefore bounded so that the double Dirichlet series above converges for $\sigma > 1$. In particular, it is holomorphic on the region $\sigma > 1,\ \tau > 1$.

If $\tau \leqslant -\varepsilon$, we appeal to the functional equation (4.19) and note that all the extra factors appearing there are holomorphic. The convexity bound yields that $L^S(w,\chi_M\beta) \ll |M|^{\frac{1}{2}-\tau}$ for $\tau < 0$, so that the series converges in the domain $\sigma+\tau > 3/2+\varepsilon$. In order to interpolate these bounds in between by the Phragmén-Lindelöf principle, it is necessary to get rid of the simple pole of $L^S(w,\chi_M\beta)$ at $w = 1$ when the character is trivial, what is done by adding the factor $w-1$. The Phragmén-Lindelöf principle therefore gives $(w-1)L^S(w,\chi_M\beta) \ll |M|^{(1-\tau)/2}$, so that the series converges in the region $\sigma + \tau/2 > 3/2+\varepsilon$. Altogether, we conclude that $(w-1)Z^S(s,w;\pi,\alpha\delta_E,\beta)$ is holomorphic in the region $R_{1,2} \subseteq \mathbf{C}^2$ made of $(s,w) \in \mathbf{C}^2$ such that

$$\begin{cases} \sigma > \frac{3}{2} - \nu & \text{if} \quad \nu \leqslant -\varepsilon, \\ \sigma > \frac{3}{2} - \frac{1}{2}\nu & \text{if} \quad -\varepsilon < \nu < 1+\varepsilon, \\ \sigma > 1 & \text{if} \quad \nu \geqslant 1+\varepsilon, \end{cases} \tag{5.5}$$

ending the proof with $R_1 = R_{1,1} \cup R_{1,2}$. □

We apply now repeatedly the functional equations and the Hartog continuation principle to get the meromorphic continuation on the whole $\mathbf{C}^2$. Introduce the functions

$$\Phi(s,w) = \prod_{P \in S} \prod_j \left(1 - \frac{\alpha\gamma_j(P)^2}{|P|^{2-2s}}\right),$$

$$\Psi(s,w) = \prod_{P \in S} \left(1 - \frac{\beta(P)^2}{|P|^{2-2w}}\right),$$

$$P(s,w) = w(w-1)\left(w+3s-\frac{3}{2}\right)\left(w+3s-\frac{5}{2}\right)(3s+2w-3).$$

Proposition 8 (Meromorphic continuation)**.** *Let*

$$\xi(s,w) = P(s,w)\Phi(s,w)\Phi(\phi(s,w))\Phi(\psi(s,w))\Phi(\psi\phi(s,w))\Psi(s,w)\Psi(\phi(s,w)).$$

Then, the completed double Dirichlet series

$$\xi(s,w)Z^S(s,w;\pi,\alpha,\beta) \tag{5.6}$$

admits an analytic continuation to $\mathbf{C}^2$.

Proof. Up to a uniformly finite linear combination, Lemma 2 ensures that we are reduced to prove that, for every classes E, E' in H_C, the completed function $\xi(s,w)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$ admits an analytic continuation to $\mathbf{C}^2$. We use the functional equations combined with the Hartog continuation principle to extend the previous domain of holomorphy R_1 of $(w-1)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$ to the whole $\mathbf{C}^2$. Indeed, applying the functional equation (4.8) transforming by ϕ, we get that

$$\Phi(s,w)\left(w+3s-\frac{5}{2}\right)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}) \tag{5.7}$$

is analytic on $\phi(R_1)$, so that adding all the completing factors we deduce that

$$\Phi(s,w)(w-1)\left(w+3s-\frac{5}{2}\right)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'}) \tag{5.8}$$

is analytic on $R_1 \cup \phi(R_1)$. The Hartog continuation principle therefore allows to analytically continue it to the convex hull R_2 of the union $R_1 \cup \phi(R_1)$.

Applying now the functional equation (4.19) transforming by ψ, we get that

$$\Psi(s,w)\Phi(\psi(s,w))w(w-1)(3s+2w-3)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$$

is analytic on $\psi(R_2)$, so that adding all the completing factors we deduce that

$$\Psi(s,w)\Phi(s,w)\Phi(\psi(s,w))w(w-1)\left(w+3s-\frac{5}{2}\right)(3s+2w-3) \\ \times Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$$

is analytic on $R_2\cup\psi(R_2)$. The Hartog continuation principle therefore allows to analytically continue it to the convex hull R_3 of the union $R_2\cup\psi(R_2)$.

Applying again the functional equation (4.8) transforming by ϕ and gathering all the extra factors, we get that $\xi(s,w)Z^S(s,w;\pi,\alpha\delta_E,\beta\delta_{E'})$ is analytic on the convex hull of $R_3\cup\phi(R_3)$, that happens to be the whole $\mathbf{C}^2$, similarly to the domain of Diaconu, Goldfeld and Hoffstein [?]. □

6. SIEVING PROCESS

6.1 SIEVING OUT SQUARES

Now it remains to show that these good analytic properties of $Z^S(s,w;\pi,\alpha,\beta)$, in particular its analytic continuation proven in Proposition 8, transfer into good analytic properties of the original double Dirichlet series without correction factors.

Introduce the squarefree part of the pure double Dirichlet series,

$$Z_\star^S(s,w;\pi,\alpha,\beta)=\sum_{\substack{D\in I(S)\\ \text{squarefree}}}\frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w}\beta(D). \tag{6.1}$$

Remark. Adding or not the correction factors is an empty question since they are by definition trivial on squarefree ideals. The very reason of their introduction is precisely the existence of square parts that need to be compensated in order to write the functional equations.

Introduce, for $r\in I(S)$,

$$Z_r^S(s,w;\pi,\alpha,\beta)=\sum_{\substack{D\in I(S)\\ D=D_0D_1^2\\ r|D_1}}\frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w}a^S(s,D,\pi,\alpha)\beta(D). \tag{6.2}$$

Proposition 9. *For $\sigma, \tau > 1$,*

$$Z_\star^S(s,w;\pi,\alpha,\beta) = \sum_{r\in I(S)} \mu(r) Z_r^S(s,w;\pi,\alpha,\beta). \tag{6.3}$$

Proof. By definition, the right hand side in (6.3) can be rewritten as

$$\sum_{r\in I(S)} \mu(r) \sum_{\substack{D\in I(S)\\ D=D_0D_1^2\\ r|D_1}} \frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w} a^S(s,D,\pi,\alpha)\beta(D). \tag{6.4}$$

The proposition reduces to a property holding more generally for any arithmetic function $f(D)$. Indeed, summing over ideals in $I(S)$,

$$\sum_r \mu(r) \sum_{\substack{D\\ r|D_1}} f(D) = \sum_D f(D) \sum_{r|D_1} \mu(r) = \sum_{\substack{D\\ \text{squarefree}}} f(D), \tag{6.5}$$

by Möbius inversion, since the sum of $\mu(r)$ reduces to selecting the D for which $D_1 = 1$, that is to say squarefree ideals. Taking $f(D)$ to be the summands in (6.4) yields the result. □

According to (6.3), we are reduced to studying double Dirichlet series of type $Z_r^S(s,w;\pi,\alpha,\beta)$ for squarefree ideals r. Let $l \in I(S)$. Introduce

$$Z_{(l)}^S(s,w;\pi,\alpha,\beta) = \sum_{\substack{D\in I(S)\\ D=D_0D_1^2\\ (D_1,l)=1}} \frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w} a^S(s,D,\pi,\alpha)\beta(D). \tag{6.6}$$

Lemma 4. *For $\sigma, \tau > 1$, and $r \in I(S)$ squarefree,*

$$Z_r^S(s,w;\pi,\alpha,\beta) = \sum_{l|r} \mu(l) Z_{(l)}^S(s,w;\pi,\alpha,\beta). \tag{6.7}$$

Proof. This relation holds more generally for any arithmetic function $f(D)$. Indeed, switching summations, we get

$$\sum_{l|r} \mu(l) \sum_{\substack{D\\ (D_1,l)=1}} f(D) = \sum_D f(D) \sum_{l|\frac{r}{(r,D_1)}} \mu(l). \tag{6.8}$$

This last relation is justified since the relations $l|r$ and $(D_1,l) = 1$ rewrite $r = (r,D_1)lk$ for an integer ideal k, that is to say l divides $r/(r,D_1)$. The result follows by Möbius inversion, for the last sum translates into the condition $r = (r,D_1)$, that is to say $r|D_1$. With $f(D)$ the summand in (6.6), this is the claimed result. □

This proposition along with the relation (6.2) shows that the squarefree pure double Dirichlet series $Z_\star^S(s,w;\pi,\alpha,\beta)$ can be expressed as a sum of the corrected double Dirichlet series $Z^S_{(l)}(s,w;\pi,\alpha,\beta)$. It is also possible to express the latter in terms of the corrected double Dirichlet series $Z^{S_l}(s,w;\pi,\alpha,\beta)$, for which many properties are known by the previous sections. This is the content of the following result.

Proposition 10. *For every characters α,β of finite order,*

$$\prod_{j=1}^{3}\prod_{P|l}\left(1-\frac{\alpha\gamma_j(P)^2}{|P|^{2s}}\right)Z^S_{(l)}(s,w;\pi,\alpha,\beta) \tag{6.9}$$

is a uniformly finite linear combination of expressions of the form

$$\sum_{l_3|l}|l_3|^{-w}\sum_{m_j|l_3}\frac{\chi_{m_1m_2m_3}\gamma_j\beta\rho(l_3)}{|m_1m_2m_3|^s}Z^{S_l}(s,w;\pi,\alpha\chi_{l_3},\beta\rho\chi_{m_1m_2m_3}). \tag{6.10}$$

Proof. This is essentially [?, Proposition 2.2] or the analogous [?, Proposition 4.14] adapted to the number field case. Since none of these articles prove the result explicitly, we provide the details here.

By definition, we have

$$Z^S_{(l)}(s,w;\pi,\alpha,\beta)=\sum_{(D_1,l)=1}\frac{L^S(s,\pi\otimes\chi_D\alpha)}{|D|^w}\beta(D)a^S(s,D,\pi,\alpha). \tag{6.11}$$

Introduce the variable $l_3=(D_0,l)$, so that l_3 is squarefree as D_0. Make the change of variables $D_0\to D_0l_3$ and $l\to ll_3$, so that

$$Z^S_{(l)}(s,w;\pi,\alpha,\beta)=\sum_{\substack{l_3|l\\ \text{squarefree}}}\sum_{(D_0D_1,l)=1}\frac{L^S(s,\pi\otimes\chi_{Dl_3}\alpha)}{|D|^w|l_3|^w}\beta(Dl_3)a^S(s,\pi,Dl_3,\alpha). \tag{6.12}$$

Making appear explicitly the local L-factor relative to the place l yields

$$L^S(s,\pi\otimes\chi_{Dl_3}\alpha)=L^{S_l}(s,\pi\otimes\chi_{Dl_3}\alpha)\prod_{P|l}\prod_{j=1}^{3}\left(1-\frac{\chi_{D_0l_3}\gamma_j(P)}{|P|^s}\right)^{-1}. \tag{6.13}$$

Introducing this expression in (6.11) and multiplying both sides by $\prod_j \prod_{P|l}(1 - \alpha\gamma_j(P)|P|^{-2s})$ yields

$$\begin{aligned}
&\prod_{i=1}^{3}\prod_{P|l}\left(1-\frac{\alpha\gamma_j(P)^2}{|P|^{2s}}\right)Z^S_{(l)}(s,w;\pi,\alpha,\beta)\\
&\quad = \sum_{\substack{l_3|l\\ \text{squarefree}}} |l_3|^{-w}\prod_j\prod_{P|l}\left(1-\alpha\gamma_j(P)^2|P|^{-2s}\right)\sum_{(D_0D_1,l)=1}\frac{L^{S,l}(s,\pi\otimes\chi_{Dl_3}\alpha)}{|D|^w}\\
&\qquad \times\prod_{P|l}\left(1-\frac{\alpha\chi_{D_0l_3}\gamma_j(P)}{|P|^s}\right)^{-1}\beta(Dl_3)a^S(s,\pi,Dl_3,\alpha).
\end{aligned}$$

By definition of the quadratic symbol, $\chi_{D_0l_3}(P)$ is zero as soon as $P|D_0l_3$, that is to say $P|l_3$ since $(D_0,l)=1$. Note also that, since l_3 is squarefree, the condition $p\nmid l_3$ rephrases as $P|l/l_3$. Moreover, since the characters χ_D are quadratic, we have for all j

$$\begin{aligned}
\prod_{p\nmid l_3}\frac{1-\alpha\gamma_j(P)^2|P|^{-2s}}{1-\alpha\chi_{D_0l_3}\gamma_j(P)|P|^{-s}} &= \prod_{p\nmid l_3}\frac{1-(\alpha\chi_{D_0l_3}\gamma_j(P)|P|^{-s})^2}{1-\alpha\chi_{D_0l_3}\gamma_j(P)|P|^{-s}}\\
&= \prod_{p\nmid l_3}\left(1+\frac{\alpha\chi_{D_0l_3}\gamma_j(P)}{|P|^s}\right).
\end{aligned}$$

By developing the involved Euler product, we get

$$\prod_{p\nmid l_3}\left(1+\frac{\alpha\chi_{D_0l_3}\gamma_j(P)}{|P|^s}\right)=\sum_{m|l/l_3}\frac{\alpha\chi_{D_0l_3}\gamma_j(m)}{|m|^s}. \tag{6.14}$$

Altogether, the expression above rewrites

$$\begin{aligned}
&\prod_{j=1}^{3}\prod_{P|l}\left(1-\frac{\alpha\gamma_j(P)^2}{|P|^{2s}}\right)Z^S_{(l)}(s,w;\pi,\alpha,\beta)\\
&\quad = \sum_{\substack{l_3|l\\ \text{squarefree}}} |l_3|^{-w}\sum_{(D_0D_1,l)=1}\frac{L^{S,l}(s,\pi\otimes\chi_{Dl_3}\alpha)}{|D|^w}\\
&\qquad \times\sum_{m_j|l/l_3}\frac{\alpha\chi_{D_0l_3}\gamma_j(m_1m_2m_3)}{|m_1m_2m_3|^s}\beta(Dl_3)a^S(s,\pi,Dl_3,\alpha).
\end{aligned}$$

By the quadratic reciprocity law (2.7), we get

$$\chi_{D_0}(m_1m_2m_3)=\eta(D_0,m_1m_2m_3)\chi_{m_1m_2m_3}(D_0). \tag{6.15}$$

By the orthogonality relations (3.12) we can select the ray classes in H_C for D as well as $m_1m_2m_3$ at the cost of a uniformly finite linear combination, and be reduced to sums of the form

$$\sum_{l_3|l} |l_3|^{-w} \sum_{m_j/l_3} \frac{\chi_{m_1m_2m_3}\gamma_j\beta\rho(l_3)}{|m_1m_2m_3|^s} \sum_{(D_0D_1,l)=1} \frac{L^{S_l}(s,\pi\otimes\chi_{Dl_3}\alpha)}{|D|^w}\beta\rho(D)a^S(s,\pi,Dl_3).$$

Recalling the property that $a^S(s,\pi,Dl_3,\alpha) = a^S(s,\pi,D,\alpha\chi_{l_3})$, see for instance [**?**, Equation (4.9)] we recognize

$$\sum_{l_3|l} |l_3|^{-w} \sum_{m_j|l_3} \frac{\chi_{m_1m_2m_3}\gamma_j\beta\rho(l_3)}{|m_1m_2m_3|^s} Z^{S_l}(s,w;\pi,\alpha\chi_{l_3},\beta\rho\chi_{m_1m_2m_3}),$$

that is the claimed result. □

6.2 VERTICAL BOUNDS

The relation (6.3) is unfortunately not a linear combination, but an infinite sum. In particular, extra bounds in the l-aspect are necessary in order to ensure convergence so that the sum remains meromorphic. This is the aim of the following and this is precisely where Chinta and Diaconu were not able generalize their results to all number fields, due to the lack of large sieve inequalities in this setting. They relied on a quadratic large sieve result due to Heath-Brown [**?**]. Recent results due to Goldmakher and Louvel [**?**] open the path to generalizing this point.

Lemma 5 (Goldmakher-Louvel)**.** *Let F be a number field and X be the set of quadratic characters χ_D with D integer ideals out of S. Let $(\lambda_D)_D$ be a sequence of complex numbers parametrized by integral ideals D of F. Then we have for every $\varepsilon > 0$ and every $M,N \geqslant 1$,*

$$\sideset{}{^\star}\sum_{|A|\leqslant M} \left| \sideset{}{^\star}\sum_{|D|\leqslant N} \lambda_D\chi_D(A)\right|^2 \ll_\varepsilon (MN)^\varepsilon(M+N) \sideset{}{^\star}\sum_{ND\leqslant N} |\lambda_D|^2, \qquad (6.16)$$

where the starred sums stands for sums restricted to squarefree ideals.

We deduce from this quadratic large sieve inequality for general number fields the following estimate on mean square of twisted central values.

Lemma 6. *Let π be a self-contragredient cuspidal automorphic representation of* GL(3) *over F. For all $\varepsilon > 0$ and character α of finite order we have, for all $X > 0$,*

$$\sum_{|D|\leqslant X} \left| L\left(\frac{1}{2},\pi\otimes\chi_D\alpha\right)\right|^2 \ll_\varepsilon X^{3/2+\varepsilon}. \qquad (6.17)$$

For the remainder of the section the bounds for $Z^{S_r}(s,w;\pi,\alpha,\beta)$ are given for $s=\frac{1}{2}$. In practice, we will need them around a small neighborhood of $s=\frac{1}{2}$, and it is straightforward to check that the bounds below still hold for s in a small compact neighborhood of $\frac{1}{2}$.

Proposition 11. *For $\tau > 5/4+\varepsilon$ and characters α,β of finite order,*

$$Z^{S_r}\left(\frac{1}{2},w;\pi,\alpha,\beta\right) \ll_\varepsilon |r|^\varepsilon. \tag{6.18}$$

Proof. Proposition 8 states that the function $Z^{S_r}\left(\frac{1}{2},w;\pi,\alpha,\beta\right)$ is analytic in w except for possible poles at $w=0$, $w=\frac{1}{2}$, $w=\frac{3}{4}$ and $w=1$. Moreover, $Z^{S_r}\left(\frac{1}{2},w;\pi,\alpha,\beta\right)$ converges by Lemma 5. Writing down partial sums and underlining that χ_D only depends on the squarefree part D_0 of D, we get

$$\begin{aligned}
&\sum_{\substack{|D|\leqslant X\\(D,r)=1}} \frac{L^{S_r}\left(\frac{1}{2},\pi,\chi_D\alpha\right)}{|D|^w}\beta(D)a^{S_r}\left(\frac{1}{2},\pi,D,\alpha\right)\\
&\quad\ll_\varepsilon |r|^\varepsilon \sum_{\substack{|D_0|\leqslant X\\(D_0,r)=1}} \frac{\left|L^{S_r}\left(\frac{1}{2},\pi\otimes\chi_{D_0}\alpha\right)\right|}{|D_0|^\tau} \sum_{\substack{|D_1|^2\leqslant X/|D|\\(D_1,r)=1}} \frac{\left|a^{S_r}\left(\frac{1}{2},\pi,D,\alpha\right)\right|}{|D_1|^{2\tau}}\\
&\quad\ll_\varepsilon |r|^\varepsilon \sum_{\substack{|D_0|\leqslant X\\(D_0,r)=1}} \frac{\left|L^{S_r}\left(\frac{1}{2},\pi\otimes\chi_{D_0}\alpha\right)\right|}{|D_0|^\tau} \sum_{(D_1,r)=1} \frac{\left|a^{S_r}\left(\frac{1}{2},\pi,D,\alpha\right)\right|}{|D_1|^{2\tau}}\\
&\quad\ll_\varepsilon |r|^\varepsilon \sum_{(D_0,r)=1} \frac{\left|L^{S_r}\left(\frac{1}{2},\pi\otimes\chi_{D_0}\alpha\right)\right|}{|D_0|^\tau}
\end{aligned}$$

and this last sum is convergent for $\tau > 5/4$ by Lemma 6. We therefore obtain for $\tau > 5/4+\varepsilon$,

$$Z^{S_r}\left(\frac{1}{2},w;\pi,\alpha,\beta\right) \ll |r|^\varepsilon \sum_{\substack{D_0\\(D_0,r)=1\\D_0 \text{ squarefree}}} \frac{\left|L\left(\frac{1}{2},\pi\otimes\alpha\chi_{D_0}\right)\right|}{|D_0|^\tau}, \tag{6.19}$$

achieving to prove the claim. ∎

Proposition 12. *For $\tau = -1/4-\varepsilon$ and α,β characters of finite order on ideals,*

$$Z^{S_l}\left(\frac{1}{2},w;\pi,\alpha,\beta\right) \ll_\varepsilon |l|^{5+\varepsilon} \sum_{\rho\in\widehat{H}_C}\sum_{D_0} \frac{\left|L\left(\frac{1}{2},\pi\otimes\chi_D\rho\right)\right|}{|D_0|^{5/4+\varepsilon}c(\rho)^{1/4}}. \tag{6.20}$$

Proof. This is analogous to [?, bound (3.7)]. First of all, we know explicit bounds on the local L-factors as a consequence of the Stirling formula, see for instance [?, Equation (3.5)]. We have for all $\sigma_1 > \sigma_2$ and large enough $|t|$, and for π an automorphic cuspidal representation on GL_n for $n \in \{1,3\}$.

$$\prod_{v \in S_\infty} \frac{L_v(\sigma_1 + it, \pi \otimes \chi_D)}{L_v(\sigma_2 - it, \pi \otimes \chi_D)} \ll (|t|+1)^{n(\sigma_1-\sigma_2)/2}. \tag{6.21}$$

Moreover, since every character appearing in the functional equations is of finite order and therefore of norm bounded by one, and the functional equations involve uniformly finite linear combinations, (4.8) and (4.19) imply the following bounds.

Lemma 7. *For an automorphic cuspidal representation* π *on* $\mathrm{GL}(n)$*, for* $n \in \{1,3\}$*,* α, β *characters on ideals,* E *a class in* H_C *prime to* S*, we have*

$$Z^{S_r}(s,w;\pi,\alpha,\beta\delta_E) \tag{6.22}$$

bounded by a uniformly finite linear combination of expressions, for $\rho \in \widehat{H}_C$*,*

$$\ll f(\alpha)^{3(\frac{1}{2}-\sigma)} \prod_{P|r/f_r(\alpha)} \prod_j \left|1 - \frac{\alpha\gamma_j(P)^2}{|P|^{2-2s}}\right|^{-1} \sum_{l_j|r/f_r(\alpha)} \frac{|\gamma_j(l_j)|}{|l_j|^{1-\sigma}} \tag{6.23}$$
$$\times \sum_{m_j|r/f_r(\alpha)} \frac{|\gamma_j(m_j)|}{|m_j|^{\sigma}} \left|Z^{S_r}(\phi(s,w);\pi,\alpha,\beta\rho\chi_\pi\chi_{l_1l_2l_3m_1m_2m_3})\right|,$$

and

$$Z^{S_r}(s,w;\pi,\alpha\delta_E,\beta) \tag{6.24}$$

bounded by a uniformly finite linear combination of expressions, for $\rho \in \widehat{H}_C$*,*

$$\ll f(\beta)^{\frac{1}{2}-\nu} \prod_{P|r/f_r(\beta)} \left|1 - \frac{\beta(P)^2}{|P|^{2-2w}}\right|^{-1} \sum_{l|r/f_r(\beta)} \frac{1}{|l|^w} \tag{6.25}$$
$$\times \sum_{m|r/f_r(\beta)} \frac{1}{|m|^{1-w}} \left|Z^{S_r}(\psi(s,w);\pi,\alpha\rho\chi_\pi\chi_{lm},\beta)\right|.$$

Note that $\psi\phi\psi\phi\psi$ maps the line $(\frac{1}{2}, \frac{5}{4}+\varepsilon+it)$ onto the line $(\frac{1}{2}, -\frac{1}{4}-\varepsilon-it)$. Because of the bound (6.18) on the former, the strategy is to apply successively the functional equations (4.8) and (4.19) in order to apply the bounds on the line $(\frac{1}{2}, \frac{5}{4}+\varepsilon+it)$, carefully gathering and bounding all the extra terms appearing in the bounds provided by Lemma 7.

The elegant matrix formulation of [?, pages 2951-2955] stands *mutatis mutandis* in our case. Indeed, the bound obtained in Proposition (6.23) is analogous to their bound (3.16), and the bound obtained in Proposition (6.25) is analogous to their bound (3.17). Since the exponents are the same everywhere, we afford not to repeat their computations. □

6.3 ANALYTIC CONTINUATION

We have now the tools required in order to analytically continue the Dirichlet series $Z_\star(s,w;\pi,\alpha,\beta)$.

Proposition 13. *The series $Z_\star(s,w;\pi,\alpha,\beta)$ has meromorphic continuation to a tube domain containing $(\frac{1}{2},1)$. More precisely, the function $\xi(s,w)Z_\star(s,w;\pi,\alpha,\beta)$ has analytic continuation in the region*

$$\left\{\sigma \geqslant \frac{1}{2}, \tau > -\frac{4}{105}\right\}. \tag{6.26}$$

Proof. We need an estimate in the l aspect in the central strip between $-1/4$ and $5/4$. We have the following bounds on correcting factors [?], for all $\nu > 1-\varepsilon$ and $\sigma \geqslant \frac{1}{2}$:

$$a(s,D,\pi,\alpha) \ll |D_1|^{15/14},$$
$$b(w,M,\pi,\beta) \ll |M|^{\varepsilon}.$$

From the explicit relation between $Z^S_{(l)}(s,w;\pi,\alpha,\beta)$ and $Z^{S_l}(s,w;\pi,\alpha,\beta)$ established in Proposition 10 and the vertical bounds obtained in the previous section we deduce that, for $\tau > 5/4+\varepsilon$,

$$Z^S_{(l)}(s,w;\pi,\alpha,\beta) \ll |l|^{-\frac{47}{14}+\varepsilon}, \tag{6.27}$$

and that, for $\tau < -1/4-\varepsilon$,

$$Z^S_{(l)}(s,w;\pi,\alpha,\beta) \ll |l|^{\frac{23}{14}+\varepsilon}. \tag{6.28}$$

Since $Z_r(s,w;\pi,\alpha,\beta)$ is a uniformly finite linear combination of functions of type $Z_{(l)}(s,w;\pi,\alpha,\beta)$ by (6.2), we get the same bounds for $Z_r(s,w;\pi,\alpha,\beta)$. By Proposition 8, $\xi(s,w)Z_r(s,w;\pi,\alpha,\beta)$ is an analytic function on the vertical strip $-1/4-\varepsilon \leqslant \tau \leqslant 5/4+\varepsilon$, so that we can apply the Phragmén-Lindelöf principle and get, for all $-1/4-\varepsilon \leqslant \tau \leqslant 5/4+\varepsilon$,

$$Z^S_r(s,w;\pi,\alpha,\beta) \ll |r|^{-5\nu+\frac{17}{21}+\varepsilon}. \tag{6.29}$$

Therefore, the sum (6.1) defining $Z_\star(s,w;\pi,\alpha,\beta)$ converges as soon as $\nu > -\frac{4}{105}$, and in particular $Z_\star(s,w;\pi,\alpha,\beta)$ is analytic in this domain as a locally uniformly convergent sum of analytic functions. □

7. RESIDUES AND FOURIER COEFFICIENTS

7.1 ANALYTIC PROPERTIES

The previous section concluded with the meromorphic continuation of the double Dirichlet series $Z_\star(s,w;\pi,\alpha,\beta)$, around the point $(\frac{1}{2},1)$. Theorem 4 will eventually follow from a precise study of the corresponding residues at $(\frac{1}{2},1)$, which are related to the Fourier coefficients of π. It is necessary to write down more precisely the local behavior around the point $(\frac{1}{2},1)$.

Let E be an element in H_C. Consider an ideal $r \in I(S)$ that is either $\mathscr{O}$ or a prime ideal. Recall that

$$Z_\star(s,w;\pi,\delta_E,\chi_r) = \sum_{\substack{D\in[E] \\ (D,r)=1 \\ \text{squarefree}}} \frac{L(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D). \tag{7.1}$$

By the decomposition of the local factors given in Lemma 3 and the constancy of $\chi_D = \chi_E$ for ideals $D \in [E]$ and places in S, we have

$$L(s,\pi\otimes\chi_D) = L_S(s,\pi\otimes\chi_E)L^S(s,\pi\otimes\chi_D), \tag{7.2}$$

so that we can rewrite

$$Z_\star(s,w;\pi,\delta_E\chi_r,1) = L_S(s,\pi\otimes\chi_E) \sum_{\substack{D\in[E] \\ (D,r)=1 \\ \text{squarefree}}} \frac{L^S(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D). \tag{7.3}$$

Call $Z_\star^S(s,w;\pi,\delta_E\chi_r,1)$ the series appearing on the right. The above justifies that we can first concentrate on studying it instead of $Z_\star(s,w;\pi,\delta_E\chi_r,1)$.

Proposition 14. *The function $\xi(s,w)Z_\star^S(s,w;\pi,\delta_E\chi_r,1)$ admits an analytic continuation in the region*

$$\left\{\sigma \geqslant \frac{1}{2}, \quad \tau > -\frac{4}{105}\right\}. \tag{7.4}$$

Proof. • Let begin with the case $r = \mathscr{O}$, so that

$$Z_\star^S(s,w;\pi,\delta_E,1) = \sum_{\substack{D\in[E] \\ \text{squarefree}}} \frac{L^S(s,\pi\otimes\chi_D)}{|D|^w}. \tag{7.5}$$

By the orthogonality relations in $\widehat{H}_C$, it is a uniformly finite linear combination of functions of type $Z_\star^S(s,w;\pi,\rho,1)$ for $\rho \in \widehat{H}_C$, so that by Proposition 13, it also admits a meromorphic continuation to $\mathbf{C}^2$.

• Let turn to the case where r is a prime ideal. Writing S_r for $S \cup \{r\}$, we get

$$Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = \sum_{\substack{D\in[E]\\ \text{squarefree}}} \frac{L^S(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D) \tag{7.6}$$

$$= \sum_{\substack{D\in[E]\\ \text{squarefree}}} \frac{L^{S_r}(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D)L_r(s,\pi\otimes\chi_D). \tag{7.7}$$

Recalling that χ_r is a quadratic character so that it only takes the two values 1 and -1 according to the parity of the power of r evaluated, we have

$$L_r(s,\pi\otimes\chi_D) = \sum_{k\geqslant 0} \frac{c\left(r^k\right)}{|r^k|^s}\chi_D\left(r^k\right) = \chi_D(r)L_{1,r}(s) + L_{2,r}(s), \tag{7.8}$$

where we defined the two partial series according to parity

$$L_{1,r}(s) = \sum_{k\geqslant 0} \frac{c\left(r^{2k+1}\right)}{|r^{2k+1}|^s} \qquad \text{and} \qquad L_{2,r}(s) = \sum_{k\geqslant 0} \frac{c\left(r^{2k}\right)}{|r^{2k}|^s}. \tag{7.9}$$

By the Blomer-Brumley bounds [?] for the Fourier coefficients of automorphic forms on $\mathrm{GL}(3)$, we have that $c(N) \ll |N|^{5/14+\varepsilon}$ so that both series $L_{1,r}(s)$ and $L_{2,r}(s)$ absolutely uniformly converge on the half-plane $\sigma > 5/14+\varepsilon$.

By the quadratic reciprocity law (2.7), since the sum is over a fixed class in H_C with representative E, we have, for all $Dw \in [E]$,

$$\chi_D(r) = \chi_r(D)\eta(E,r). \tag{7.10}$$

Thus, (7.7) rewrites

$$\begin{aligned} Z_\star^S(s,w;\pi,\delta_E\chi_r,1) &= \eta(E,r)L_{1,r}(s) \sum_{\substack{D\in[E]\\ (D,r)=1\\ \text{squarefree}}} \frac{L^{S_r}(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D) \\ &\quad + L_{2,r}(s) \sum_{\substack{D\in[E]\\ (D,r)=1\\ \text{squarefree}}} \frac{L^{S_r}(s,\pi\otimes\chi_D)}{|D|^w} \\ &= \eta(E,r)L_{1,r}(s)Z_\star^{S_r}(s,w;\pi,\delta_E,1) + L_{2,r}(s)Z_\star^{S_r}(s,w;\pi,\delta_E\chi_r,1). \end{aligned}$$

By the analytic continuation of the functions $Z_\star^S(s,w;\pi,\alpha,\beta)$ around $(\frac{1}{2},1)$ obtained in Proposition 13 we conclude that $Z_\star^S(s,w;\pi,\delta_E\chi_r,1)$ admits a meromorphic continuation around $(\frac{1}{2},1)$, and more precisely that the same double Dirichlet series completed by the factor $\xi(s,w)$ admits analytic continuation on the desired region. □

7.2 COMPUTATION OF RESIDUES AT $w=1$

We aim at understanding the residue

$$\operatorname*{Res}_{w=1} Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = \lim_{w\to 1}(w-1)Z_\star^S(s,w;\pi,\delta_E\chi_r,1). \tag{7.11}$$

Proposition 15. *We have*

$$Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = h_C^{-1}\sum_{\rho\in\widehat{H}_C}\rho^{-1}(E)\sum_{N\in I(S)}\frac{c(N)}{|N|^s}\eta(E,N_0)L(w,\rho\chi_r\chi_{N_0}). \tag{7.12}$$

Proof. The series $Z_\star^S(s,w;\pi,\delta_E\chi_r,1)$ absolutely converges for $\sigma \geqslant \frac{1}{2}$. In this region, we can therefore interchange the summation and recognize GL(1) L-functions for which residues are explicitly computable. More precisely, when r is a prime ideal in $I(S)$,

$$\begin{aligned}
Z_\star^S(s,w;\pi,\delta_E\chi_r,1) &= \sum_{\substack{D\\(D,r)=1\\\text{squarefree}}}\frac{L^S(s,\pi\otimes\chi_D)}{|D|^w}\chi_r(D)\\
&= \sum_{\substack{D\\(D,r)=1\\\text{squarefree}}}\frac{\chi_r(D)}{|D|^w}\sum_{N\in I(S)}\frac{c(N)\chi_D(N)}{|N|^s}\\
&= \sum_N\frac{c(N)}{|N|^s}\sum_{\substack{D\\(D,r)=1\\\text{squarefree}}}\frac{\chi_r(D)\chi_D(N)}{|D|^w}.
\end{aligned}$$

Formally, we want to apply the quadratic reciprocity law to recognize the rightmost series as a GL(1) L-function. More precisely, decompose $N=N_0N_1^2N_2^2$ with N_0 the squarefree part of N and N_1 such that $P|N_1 \implies P|N_0$. In other words, N_1 is the square part of N with the same prime factors than N_0. If $(D,N_2)\neq 1$, since D is squarefree, we have $\chi_D(N_2)=0$ and therefore $\chi_D(N)=\chi_D(N_2)^2\chi_D(N_0N_1^2)=0$. We thus assume from now on that $(D,N_2)=1$. If $(D,N_0)=1$ we have, by the quadratic reciprocity law,

$$\chi_D(N)=\chi_D(N_0)=\eta(E,N_0)\chi_{N_0}(D). \tag{7.13}$$

If $(D,N_0)\neq 1$, we have

$$\chi_D(N)=\chi_D(N_0)=0=\eta(E,N_0)\chi_{N_0}(D), \tag{7.14}$$

so that the relation is also true. We can therefore write, using the orthogonality relations,

$$Z_{\star}^{S}(s,w;\pi,\delta_E\chi_r,1) = \sum_N \frac{c(N)}{|N|^s} \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \eta(E,N_0)\frac{\chi_r(D)\chi_{N_0}(D)}{|D|^w}$$

$$= \sum_N \frac{c(N)}{|N|^s} h_C^{-1} \sum_{\rho\in\widehat{H}_C} \rho^{-1}(E)\eta(E,N_0) \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\rho(D)\chi_r(D)\chi_{N_0}(D)}{|D|^w}.$$

We recognize the the innermost sum is the GL(1) L-function $L(w,\rho\chi_r\chi_{N_0})$. □

We can now relate the residue we are interested in in terms of residues of GL(1) L-functions and of the Dedekind zeta function associated to the number field F.

Lemma 8. *We have*

$$\begin{aligned}(1+|r|^{-1})^{-1} \sum_{(N_2,r)=1} \frac{c(N_2^2)}{|N_2^2|^s} \prod_{P|N_2}\left(1+\frac{1}{|P|}\right)^{-1} \\ = \frac{L_{1,r}(s)}{|r|^{-1}+L_{2,r}(s)} \sum_N \frac{c(N^2)}{|N^2|^s} \prod_{P|N}\left(1+\frac{1}{|P|}\right).\end{aligned} \tag{7.15}$$

Proof. This is exactly [**?**, Equation (4.7)]. □

Proposition 16. *Introduce $A(F)$ the residue at 1 of the Dedekind zeta function ζ_F associated to F. Then we have, for all character ϕ on $I(S)$,*

$$\lim_{w\to 1}(w-1)\zeta_F(2w) \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = \begin{cases} \displaystyle\prod_{\substack{p\notin S \\ \text{or } p\nmid Nr}} \left(1+\frac{1}{|P|^w}\right)^{-1} A(F) & \text{if} \quad \psi = 1; \\ 0 & \text{otherwise.}\end{cases} \tag{7.16}$$

Proof. Recall that the Dedekind zeta function attached to F is defined by

$$\zeta_F(w) = \prod_p \left(1-\frac{1}{|P|^w}\right)^{-1}. \tag{7.17}$$

Moreover, we have the Euler product expansion in the domain of convergence $\tau > 1$,

$$\sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = \prod_{\substack{p\notin S \\ p\nmid Nr}} \left(1+\frac{\psi(P)}{|P|^w}\right). \tag{7.18}$$

We therefore have the relation

$$\zeta_F(2w) \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = \prod_{\substack{p\in S \\ P|rN}} \left(1-|\frac{1}{|P|^{2w}}\right)^{-1} \prod_{\substack{p\notin S \\ \text{or } p\nmid Nr}} \left(1-\frac{\psi(P)}{|P|^w}\right)^{-1}. \tag{7.19}$$

For $\psi \neq 1$, we recognize the $\mathrm{GL}(1)$ partial L-series

$$\prod_{\substack{p\notin S \\ \text{or } p\nmid Nr}} \left(1-\frac{\psi(P)}{|P|^w}\right)^{-1} = L^{S,Nr}(w,\psi), \tag{7.20}$$

which is therefore holomorphic on $\tau > 0$. So there is no pole on this half-plane containing $w = 1$, and we therefore deduce

$$\lim_{w\to 1}(w-1)\zeta_F(2w) \sum_{\substack{D \\ (D,rn_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = 0. \tag{7.21}$$

For $\psi = 1$, we have $\psi_{r,N_0} = \rho\chi_r\chi_{N_0} = 1$, so that $N_0 = r$. We can thus write $N = r^{2k+1}N_2^2$ for a certain $k \geqslant 0$ and $N_1 \in I(S)$ prime to r. Explicitly we can therefore compute

$$\zeta_F(2w) \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = \prod_{\substack{p\notin S \\ \text{or } p\nmid Nr}} \left(1+\frac{1}{|P|^w}\right)^{-1}, \tag{7.22}$$

so that,

$$\lim_{w\to 1}(w-1)\zeta_F(2w) \sum_{\substack{D \\ (D,rN_2)=1 \\ \text{squarefree}}} \frac{\psi(D)}{|D|^w} = 0 = \prod_{\substack{p\notin S \\ \text{or } p\nmid Nr}} \left(1+\frac{1}{|P|^w}\right)^{-1} A(F). \tag{7.23}$$

This proves the claim. □

Proposition 17. *We have*

$$\operatorname*{Res}_{w=1} Z_\star^S(s,w;\pi,\delta_E\chi_r,1) =$$

$$\frac{A(F)}{h_C}\frac{\eta(E,r)}{\zeta_F(2)}\prod_{p\in S}\left(1+\frac{1}{|P|}\right)^{-1}\sum_N \frac{c(N^2)}{|N^2|^s}\prod_{P|N}\left(1+\frac{1}{|P|}\right)^{-1}\frac{L_{1,r}(s)}{\frac{1}{|r|}+L_{2,r}(s)}.$$

Coming back to $Z_\star^S(s,w;\pi,\alpha,\beta)$ we have, by switching summations,

$$\lim_{w\to 1}\zeta_F(2w)(w-1)Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = h_C^{-1}\sum_{\rho\in\widehat{H}_C}\rho^{-1}(E)\sum_N\frac{c(N)}{|N|^s}\eta(E,N_0)\lim_{w\to 1}(w-1)\zeta_F(2w)\sum_{\substack{D\\(D,rN_2)=1\\ \text{squarefree}}}\frac{\rho\chi_r\chi_{N_0}(D)}{|D|^w}.$$

It is possible to switch the limit and the summation. Indeed, for $\rho=1$, every quantity is independent of N, except the condition $(D_2,N)=1$, but $\prod_{P|N}(1+|P|^{-w})^{-1}\ll|N|^\varepsilon$ uniformly when $w\to 1$. Thus for $\sigma>5/14+\varepsilon$, it is possible to interchange. For $\rho\neq 1$, we can explicitly bound in $|N|$ by the Phragmén-Lindelöf principle, and the whole sum will be uniformly controlled by $\sum c(N)|N|^{-s}$ which uniformly converges for σ large enough. Finally, we get by the above proposition

$$\begin{aligned}
&\lim_{w\to 1}\zeta_F(2w)(w-1)Z_\star^S(s,w;\pi,\delta_E\chi_r,1)\\
&= h_C^{-1}\sum_{\rho\in\widehat{H}_C}\rho^{-1}(E)\sum_N\frac{c(N)}{|N|^s}\eta(E,N_0)\lim_{w\to 1}(w-1)\zeta_F(2w)\sum_{\substack{(D,rN_2)=1\\ \text{squarefree}}}\frac{(\rho\chi_r\chi_{N_0})(D)}{|D|^w}\\
&= h_C^{-1}\eta(E,r)\sum_{\substack{N_2\\ k\geqslant 0\\ (N_2,r)=1}}\frac{r^{2k+1}N_2^2}{|N_2|^s}\prod_{p\in S}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+|r|^{-1}\right)^{-1}\prod_{P|N_2}\left(1+\frac{1}{|P|}\right)^{-1}A(F)\\
&= h_C^{-1}A(F)\eta(E,r)\prod_{p\in S}\left(1+\frac{1}{|P|}\right)^{-1}L_{1,r}(s)\sum_{N_2}\frac{c(N_2^2)}{|N_2^2|^s}\prod_{P|N_2}\left(1+\frac{1}{|P|}\right)^{-1}.
\end{aligned}$$

Finally we get, dividing by the known residues above,

$$\operatorname*{Res}_{w=1}Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = \frac{A(F)}{h_C}\frac{\eta(E,r)}{\zeta_F(2)}\prod_{p\in S}\left(1+\frac{1}{|P|}\right)^{-1}\sum_N\frac{c(N^2)}{|N^2|^s}\prod_{P|N}\left(1+\frac{1}{|P|}\right)^{-1}\frac{L_{1,r}(s)}{\frac{1}{|r|}+L_{2,r}(s)}.$$

This is an explicit expression of the residue in terms of places in S. □

In order to handle the computations to come, we need a technical results concerning the quantities appearing above.

Lemma 9. *We have for $\sigma>5/14$,*

$$\sum_N\frac{c(N^2)}{|N^2|^s}\prod_{P|N}\left(1+\frac{1}{|P|}\right)^{-1}=L^S(2s,\pi,\mathrm{sym}^2). \tag{7.24}$$

Proof. This is [**?**, Proposition 4.2]. □

Lemma 10. *The functions $L_{1,r}(s)$ and $L_{2,r}(s)$ are rational fractions of the Satake parameters $\gamma_1(P)$, $\gamma_2(P)$, $\gamma_3(P)$ and the Fourier coefficients and $a_\pi(r)$ of π. More precisely,*

$$L_{1,r}(s) = \frac{a_\pi(r) + r^{-2s}}{r^s} \prod_{j=1}^{3} \left(1 - \frac{\gamma_j(r)^2}{|r|^{2s}}\right)^{-1}$$

$$L_{2,r}(s) = \frac{1 + a_\pi(r) r^{-2s}}{r^s} \prod_{j=1}^{3} \left(1 - \frac{\gamma_j(r)^2}{|r|^{2s}}\right)^{-1}.$$

Moreover, these functions are monotones for large enough r.

Proof. This is [**?**, Lemma 2.3 and Lemma 5.1]. □

Coming back to the pure double Dirichlet series $Z_\star(s,w;\pi,\delta_E\chi_r,1)$, recall that taking aside the places of S we can write

$$Z_\star(s,w;\pi,\delta_E\chi_r,1) = L_S(s,\pi\otimes\chi_E) Z_\star^S(s,w,\pi,\delta_E\chi_r,1), \tag{7.25}$$

so that in particular taking the residues at $w=1$, we get

$$\operatorname*{Res}_{w=1} Z_\star(s,w;\pi,\delta_E\chi_r,1) = L_S(s,\pi\otimes\chi_E) \cdot \operatorname*{Res}_{w=1} Z_\star^S(s,w,\pi,\delta_E\chi_r,1). \tag{7.26}$$

We can summarize the explicit dependencies on r of these residues.

Proposition 18. *We have, for every prime ideal r,*

$$\operatorname*{Res}_{w=1} Z_\star(s,w;\pi,\delta_E,1) = R_1(s,\pi,E),$$
$$\operatorname*{Res}_{w=1} Z_\star(s,w;\pi,\delta_E\chi_r,1) = R_1(s,\pi,E) R_r(s,\pi,E).$$

where, for a certain Euler product $T(s)$ absolutely convergent for $\sigma > 1/5$,

$$R_1(s,\pi,E) = L_S(s,\pi\otimes\chi_E) L^S(2s,\pi,\mathrm{sym}^2) T(s), \tag{7.27}$$

$$R_r(s,\pi,E) = \eta(E,r)(1+|r|^{-1}) \frac{L_{1,r}(s)}{\frac{1}{|r|} + L_{2,r}(s)}. \tag{7.28}$$

Proof. Proposition 17 states the explicit formula

$$\operatorname*{Res}_{w=1} Z_\star^S(s,w;\pi,\delta_E\chi_r,1) =$$
$$\frac{A(F)}{h_C} \frac{\eta(E,r)}{\zeta_F(2)} \prod_{p\in S} \left(1 + \frac{1}{|P|}\right)^{-1} \sum_N \frac{c(N^2)}{|N^2|^s} \prod_{P|N} \left(1 + \frac{1}{|P|}\right)^{-1} \frac{L_{1,r}(s)}{\frac{1}{|r|} + L_{2,r}(s)},$$

so that the proposition is a straightforward application of Lemmata 8 and 9. □

Remark. The partial L-factor $L_S(s,\pi\otimes\chi_E)$ is omitted in [**?**]. Even though it is present in their expression (4.9), it should be added back from (4.12) until the end of the paper. In particular, it modifies the definition of their $C_M(\pi)$ in (4.21). This is not a critical issue since the partial L-factor $L_S(s,\pi\otimes\chi_E)$ never vanishes by the Euler product expression and the known bounds towards the Ramanujan conjecture. We are grateful to Adrian Diaconu for having clarified this fact to us.

Coming back to the local behavior at $(\frac{1}{2},1)$, recall that the analytic continuation stated by Proposition 14 implies the local development

$$Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = \frac{A(s)}{w-1} + \frac{B(s)}{w+3s-3/2} + H(s,w), \tag{7.29}$$

where $H(s,w)$ is a holomorphic function around the point $(\frac{1}{2},1)$.

7.3 LOCAL ANALYSIS

7.3.1 • CASE OF GELBART-JACQUET LIFTS

Suppose now that π is an automorphic cuspidal self-contragredient representation of $\mathrm{GL}(3)$ with trivial central character, and that π is a Gelbart-Jacquet lift.

Proposition 19. *For π a Gelbart-Jacquet lift, there is a ray class E such that*

$$\lim_{s\to\frac{1}{2}}\lim_{w\to 1}(w-1)\left(s-\frac{1}{2}\right)Z_\star(s,w;\pi,\delta_E\chi_r,1) = 2R_1\left(\frac{1}{2},\pi\right)R_r\left(\frac{1}{2},\pi\right). \tag{7.30}$$

Proof. By the results of [?], this is equivalent to $L^S(s,\pi,\mathrm{sym}^2)$ having a pole at $s=1$. Moreover, the partial L-factor $L_S(s,\pi\otimes\chi_E)$ never vanishes by the Euler product expression and the known bounds towards the Ramanujan conjecture. Then (7.27) implies that $R_1(s,\pi,E)$ has a pole at $s=1/2$. We can therefore write

$$A(s) = \frac{A_0}{s-\frac{1}{2}} + A_1(s),$$
$$B(s) = \frac{B_0}{s-\frac{1}{2}} + B_1(s),$$

where A_1 and B_1 are holomorphic functions around $s=\frac{1}{2}$. This allows to refine the local expansion (7.29) into

$$\begin{aligned} Z_\star^S(s,w;\pi,\delta_E\chi_r,1) = {} & \frac{A_0}{(w-1)(s-\frac{1}{2})} + \frac{A_1(s)}{w-1} \\ & + \frac{B_0}{(w+3s-\frac{3}{2})(s-\frac{1}{2})} + \frac{B_1(s)}{w+3s-\frac{3}{2}} + H(s,w). \end{aligned} \tag{7.31}$$

Now, we use the functional equation

$$Z_\star^S(s,w;\pi,\delta_E,W\chi_r) = \left[\frac{\varepsilon(s,\pi\otimes\chi_E)}{|E_0|^{3(1/2-s)}}\right]\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E)}{L_v(s,\pi\otimes\chi_E)}Z_\star^S(\phi(s,w);\pi,\delta_E,W\chi_r). \tag{7.32}$$

By [?, Theorem 0.3 (ii)], there are infinitely many character twists such that $L^S(s,\pi\otimes\chi_E)$ does not vanish at $s=1/2$. Choose such a ray class E, so that $M(1/2,E)=1$. Applying the functional equation above to (7.31) yields

$$\begin{aligned} Z_\star^S(s,w;\pi,\delta_E,W\chi_r) &= \frac{A_0}{(w-1)(s-\frac12)} + \frac{A_1(s)}{w-1} \\ &\quad + \frac{B_0}{(w+3s-\frac32)(s-\frac12)} + \frac{B_1(s)}{w+3s-\frac32} + H(s,w) \\ &= M(s,E)\Bigg[-\frac{A_0}{(w+3s-\frac52)(s-\frac12)} + \frac{A_1(1-s)}{w+3s-\frac52} \\ &\quad -\frac{B_0}{(w+6s-3)(s-\frac12)} + \frac{B_1(1-s)}{w+6s-3} + H(\phi(s,w))\Bigg], \end{aligned}$$

so that in particular we deduce, by taking the residues at $s=\frac12$,

$$A_0 = B_0.$$

Taking the residue at $s=\frac12$ into the expansion (7.31), we get by Proposition 18 and adding back the local factor $L_S(s,\pi\otimes\chi_E)$,

$$\operatorname*{Res}_{s=1/2} Z_\star(s,w,\pi,\delta_E\chi_r,1) = \frac{2R_1(\frac12,\pi)R_r(\frac12,\pi)}{w-1} + H\left(\frac12,w\right), \tag{7.33}$$

and taking the residue at $w=1$ leads to

$$2R_1\left(\frac12,\pi,E\right)R_r\left(\frac12,\pi,E\right), \tag{7.34}$$

as claimed. □

Note in particular that $R_1(1/2,\pi,E)$ is nonzero. Indeed, there is a simple pole at $s=1/2$ for $L^S(s,\pi,\mathrm{sym}^2)$ so that the corresponding residue is nonzero ; moreover the partial L-factor $L_S(s,\pi\otimes\chi_E)$ does not vanish at $s=1/2$ by the Euler product expression and the known bounds towards the Ramanujan conjecture.

7.3.2 • CASE OF NON-GELBART-JACQUET LIFTS

Assume π is an automorphic cuspidal self-contragredient representation of $\mathrm{GL}(3)$ with trivial central character, and that π is not a Gelbart-Jacquet lift.

Proposition 20. *If π a non-Gelbart-Jacquet lift and if there is a ray class E such that $L(1/2,\pi\otimes\chi_E)\neq 0$, then*

$$\lim_{w\to 1}(w-1)Z_\star\left(\frac{1}{2},w;\pi,\delta_E\chi_r,1\right)=2R_1\left(\frac{1}{2},\pi,E\right)R_r\left(\frac{1}{2},\pi,E\right). \tag{7.35}$$

Proof. By the results of [?], this is equivalent to $L^S(s,\pi,\mathrm{sym}^2)$ having no pole at $s=1$. Since the partial L-factor $L_S(s,\pi\otimes\chi_E)$ is a finite Euler product, it also has no pole at $s=1/2$. Then (7.27) implies that $R_1(s,\pi,E)$ has no pole at $s=1/2$.

Now, we use the functional equation

$$Z_\star^S(s,w;\pi,\delta_E,W\chi_r)=\left[\frac{\varepsilon(s,\pi\otimes\chi_E)}{|E_0|^{3(1/2-s)}}\right]\prod_{v\in S}\frac{L_v(1-s,\pi\otimes\chi_E)}{L_v(s,\pi\otimes\chi_E)}Z_\star^S(\phi(s,w);\pi,\delta_E,W\chi_r). \tag{7.36}$$

By the non-vanishing assumption, we deduce that $M(1/2,E)=1$. Applying the functional equation above to (7.29) therefore yields

$$\begin{aligned}Z_\star^S(s,w;\pi,\delta_E,W\chi_r)&=\frac{A(s)}{w-1}+\frac{B(s)}{w+3s-3/2}+H(s,w)\\&=M(s,E)\left[\frac{A(1-s)}{w+3s-3/2}+\frac{B(1-s)}{w-1}+H(\phi(s,w))\right],\end{aligned}$$

so that in particular we deduce

$$B(s)=M(s,E)A(1-s). \tag{7.37}$$

Letting $s\to 1/2$ in (7.29) yields

$$Z_\star\left(\frac{1}{2},w;\pi,\delta_E\chi_r,1\right)=\frac{2R_1\left(\frac{1}{2},\pi,E\right)R_r\left(\frac{1}{2},\pi,E\right)}{w-1}+H\left(\frac{1}{2},w\right), \tag{7.38}$$

so that taking the residue at $w=1$ yields the result. □

We will need to know that $R_1(s,\pi,E)$ does not vanish at $s=\frac{1}{2}$ for the final argument. We summarize sufficient conditions for this in the following lemma.

Lemma 11. *If π is not a Gelbart-Jacquet lift and if there is a class E such that*

- $L(1/2,\pi\otimes\chi_E)\neq 0$,
- *if π is not a Gelbart-Jacquet lift, assume moreover that $L^S(1,\mathrm{sym}^2,\pi)\neq 0$,*

then $R_1(1/2,\pi,E)\neq 0$.

Proof. This is a straightforward consequence of (7.27). □

7.4 PROOF OF THEOREM 5

Let π a cuspidal automorphic representation of GL(3) that is not a Gelbart-Jacquet lift. In particular, that implies that $L(s,\pi,\mathrm{sym}^2)$ has no pole at $s=1$. For all $w\neq 1$ near 1, we can take $s=1/2$ and recall the local expansion proven above : there is an analytic function $H(w)$ around 1 such that

$$Z^S\left(\frac{1}{2},w;\pi,\delta_E,1\right)=\frac{2R_1\left(\frac{1}{2},\pi,E\right)}{w-1}+H\left(\frac{1}{2},w\right). \tag{7.39}$$

Assume that there is a class E such that $L^S(1/2,\pi\otimes\chi_E)\neq 0$. By Lemma 11, $R_1(1/2,\pi,E)$ does not vanish. In particular, $Z^S(1/2,w;\pi,\delta_E,1)$ has a pole at $w=1$.

We now consider the following smoothed truncated version of the double Dirichlet series. For any $x>0$, let

$$I(x)=\sum_{D\in I(S)} L^S\left(\frac{1}{2},\pi\otimes\chi_D\right)\psi(D)a^S\left(\frac{1}{2},D\right)e^{-|D|/x}. \tag{7.40}$$

Recall the expression, following [?, Theorem 3.8],

$$e^{-1/x}=\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\Gamma(x)x^w\mathrm{d}w, \tag{7.41}$$

so that in particular we can write an integral formulation of $I(x)$ given by

$$I(x)=\frac{1}{2i\pi}\int_{(2)}Z^S(1/2,w,\psi)\Gamma(x)x^w\mathrm{d}w. \tag{7.42}$$

We can therefore move the vertical integration line until $\Re(w)=-3/4-\varepsilon$ where the integral converges absolutely and contributes as an error term of order $x^{-3/4-\varepsilon}$ by the bounds on Γ and the vertical estimates provided in Section 6.2. In this process, we pick the residues corresponding to the poles of $Z^S(1/2,w;\pi,\psi)$ at 1 and $3/4$ and the pole of Γ at 0. The most significant contribution comes from the pole at 1 and it yields a contribution of

$$\operatorname*{Res}_{w=1} Z^S(1/2,w;\pi,\psi)\Gamma(x)x^w=A_0x. \tag{7.43}$$

By a theorem due to [?], $L^S(1,\pi,\mathrm{sym}^2)$ does not vanish. In particular, by Proposition 17 we get that

$$R_1\left(\frac{1}{2},\pi,E\right)=L_S(s,\pi\otimes\chi_E)L^S(1,\pi,\mathrm{sym}^2)\neq 0. \tag{7.44}$$

However, by the bound on the correcting factor $a^S(1/2,D)\ll|D_1|^{5/7+\varepsilon}$, the contribution of a $|D|\leqslant x$ can only be of size

$$\begin{aligned} L^S\left(\frac{1}{2}, \pi\otimes\chi_D\right) a^S\left(\frac{1}{2}, D\right) &\ll |D_0|^{3/4+\varepsilon}|D_1|^{5/7+\varepsilon} \\ &\ll |D_0 D_1^2|^{5/14+\varepsilon} D_0^{3/4-5/14} \\ &\ll x^{3/4+\varepsilon}. \end{aligned}$$

Therefore, a finite number of such L-factors $L^S(1/2, \pi\otimes\chi_D)$ with $|D| \leqslant x$ cannot alone contribute to a size $A_0 x$. We deduce that there should be infinitely many non-vanishing of the L-factors $L^S(1/2, \pi\otimes\chi_D)$.

7.5 PROOF OF THEOREM 4

Let π and π' be two automorphic representations as in the statement of the theorem, and E an ideal ray class as in the assumptions of Theorem 4. Let S be large enough in order to the local components of π and π' to be both unramified and principal series at the places outside S. Suppose moreover that there is a nonzero constant κ such that

$$L\left(\frac{1}{2}, \pi\otimes\chi_D\right) = \kappa\cdot L\left(\frac{1}{2}, \pi'\otimes\chi_D\right), \tag{7.45}$$

for every $D\in I(S)$. By summing over D in the same class than $E\in H_C$, we deduce the relation between the associated double Dirichlet series in the region of convergence, for every r prime or equal to $\mathscr{O}$,

$$Z_\star\left(\frac{1}{2}, w, \pi, \delta_E\chi_r, 1\right) = \kappa\cdot Z_\star\left(\frac{1}{2}, w, \pi', \delta_E\chi_r, 1\right). \tag{7.46}$$

By Proposition 19 or 20 depending on whether or not π is a Gelbart-Jacquet lift we get, for $r=\mathscr{O}$,

$$R_1\left(\frac{1}{2}, \pi, E\right) = \kappa\cdot R_1\left(\frac{1}{2}, \pi, E\right), \tag{7.47}$$

and, for r a prime ideal,

$$R_1\left(\frac{1}{2}, \pi, E\right) R_r\left(\frac{1}{2}, \pi, E\right) = \kappa\cdot R_1\left(\frac{1}{2}, \pi', E\right) R_r\left(\frac{1}{2}, \pi', E\right). \tag{7.48}$$

Since $\kappa\neq 0$ and $R_1\left(\frac{1}{2}, \pi, E\right)$ is nonzero by Lemma 11, it follows that for every prime ideal r in $I(S)$,

$$R_r\left(\frac{1}{2}, \pi, E\right) = R_r\left(\frac{1}{2}, \pi', E\right). \tag{7.49}$$

By Lemma 10, these quantities are function of the associated Fourier coefficients $a_\pi(r)$, more precisely we get for every prime ideal r in $I(S)$,

$$h_r(a_\pi(r)) = h_r(a_{\pi'}(r)), \tag{7.50}$$

for explicit functions h_r that are monotone for large enough r. In particular they are injective and the above equality yields that π and π' have same Fourier coefficients for large enough r. By the strong multiplicity one theorem, we conclude that $\pi \cong \pi'$. □

SCHOOL OF MATHEMATICS (ZHUHAI)
ZHUHAI CAMPUS, SUN YAT-SEN UNIVERSITY
TANGJIAWAN, ZHUHAI, GUANGDONG, 519082, CHINA (PRC)

Email address: lesesvre@math.cnrs.fr