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Newforms for odd orthogonal groups

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ABSTRACT

I introduce a new notion of newforms for split special odd orthogonal groups which generalizes that of classical newforms by defining a family of compact subgroups of their adelic groups.

1. Introduction

The theory of newforms has long been a fascinating subject in number theory. It originated from the Shimura–Taniyama Conjecture (or the Modularity Theorem), which associates elliptic curves over $\mathbb{Q}$ with classical holomorphic modular forms. Weil refined the conjecture and gave a precise description on such a modular form whose level equals the conductor of the elliptic curve. It comes naturally to ask if such a refinement exists for general motives; namely, if in the conjectural Langlands program we can also specify

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a particular automorphic form in the automorphic representation associated with the motive.

After his 60th birthday conference in 2010, Gross [8] wrote a letter to Serre on a conjectural construction of such an automorphic form on special odd orthogonal group to a pure symplectic motive. He proposed the existence of a distinguished line in each generic local representation. It was verified that this notion generalized the newforms for GL_2 introduced by Atkin–Lehner and Li in early 70s. He proved that such a line exists at the real place assuming the representation is a generic limit discrete series. He discussed a suggestion on the finite places by Brumer which is later proved by the author as her PhD thesis under his supervision.

In this article we review the classical theory for $\mathrm{PGL}_2\,(\simeq \mathrm{SO}_3)$ and present a conjectural framework for newforms of odd special orthogonal group with which symplectic motives are associated in the Langlands program, and describe corresponding local result in a real place and finite places.

2. Atkin–Lehner–Li theory

Consider the space of classical holomorphic cusp forms $S_k(\Gamma_0(N))$ of level N weight k. The Hecke algebra consists of averaging operators on $S_k(\Gamma_0(N))$ generated by the characteristic functions T_n of the double coset

$$\Gamma_0(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \Gamma_0(N)$$

for all $n \in \mathbb{N}$. Atkin and Lehner [1] showed that there exists a basis of $S_k(\Gamma_0(N))$ consisting of simultaneous eigenforms of T_n for all $(n, N) = 1$. Among these the newforms, opposed from oldforms, can be shown to be eigenforms of T_n for all n, and each corresponds to a unique collection of eigenvalues $\{a_n\}_{n\in\mathbb{N}}$. The eigenvalue a_n of T_n agrees with the n-th Fourier coefficient for an eigenform. The set of Fourier coefficients uniquely determines the holomorphic newform.

These conclude to a characterization of holomorphic newforms that among all nonzero eigenforms with the same collection of eigenvalues a_p for all prime number p, $(p, N) = 1$, the holomorphic newforms have the smallest level and the constant Fourier coefficients are nonzero and normalized to 1. The characterization automatically implies the sets of Fourier coefficients uniquely determine the holomorphic newforms and these newforms are eigenforms for all Hecke operators T_n, $n \in \mathbb{N}$.

Whenever a holomorphic newform f of weight k level N is given, the Fourier coefficients a_n's of f satisfy good recurrence relations gotten from the action of the Hecke operators. The theory of Atkin–Lehner [1] and Li [12] gives that the L-function

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

associated with f has the following analytic properties. Firstly, it has meromorphic continuation to the whole complex s-plane. Secondly, it admits an Euler product expansion

$$L(s,f)=\prod_p(1-a_pp^{-s}+1_N(p)p^{k-1-2s})^{-1},\quad \Re(s)>\frac{k}{2}+1.$$

(1_N is the Dirichlet character modulo N.) And thirdly, its completed L-function $\Lambda(s,f)=(2\pi)^{-s}\Gamma(s)L(s,f)$ satisfies a functional equation

$$\Lambda(k-s,f)=w(f)N^{\frac{k}{2}-s}\Lambda(s,f)$$

for some sign $w(f)\in\{\pm 1\}$ depending on the newform f.

An elliptic curve E over $\mathbb{Q}$ of conductor N induces a 2-dimensional ℓ-adic Galois representation on its Tate module $V_\ell(E)$ which is Frobenius semisimple and the inertia group acts continuously. The corresponding Artin L-function is the Hasse–Weil L-function $L(s,E)$ of the elliptic curve E such that

$$L(s,E)=\prod_p(1-a_p(E)p^{-s}+1_N(p)p^{1-2s})^{-1},\quad \Re(s)>2,$$

and has meromorphic continuation to the whole s-plane where $a_p(E)$ is the arithmetic invariant given by solution counts in $\mathbb{Z}/p\mathbb{Z}$. The Shimura–Taniyama–Weil Conjecture (the Modularity Theorem) shows there is a unique holomorphic newform $f=f_E$ of weight 2 such that

$$L(s,f)=L(s,E).$$

The holomorphic newform f_E attached to E has level N with eigenvalue $a_p(E)$ of T_p for all prime p. The automorphic representation $\pi_f=\otimes'_\nu\pi_{f,\nu}$ of $\mathrm{PGL}_2(\mathbb{A})$ generated by the associated automorphic form ϕ_f is cuspidal and has Artin conductor N. At the real place, $\pi_{f,\infty}$ is a discrete series of weight 2. At each finite place ν, $\pi_{f,\nu}$ is a generic representation.

3. Symplectic motives

Serre proposed a question asking that if there is a similar conjectural correspondence from symplectic motives to automorphic forms of orthogonal groups like the Shimura–Taniyama–Weil Conjecture in the elliptic curve case. For example, given an abelian variety A over $\mathbb{Q}$ of dimension n, the first cohomology $H^1(A)$ is a pure motive of weight 1 and rank $2n$ in the sense Deligne introduced in his Corvallis note [4]. Equivalently, one can look at the first homology group $H_1(A)$ which is a pure motive of weight -1 and rank $2n$. (The motive $H^1(A)(1)$ is isomorphic to the dual $H_1(A)$ of $H^1(A)$ by the pairing

$H^1(A) \times H^1(A) \to \mathbb{Q}(-1)$ [4].) The Weil pairing on $H_1(A)$ is nondegenerate and alternating which makes $H_1(A)$ a symplectic motive. The motivic L-function of $M = H_1(A)$ is

$$L(M,s) = \prod_p \det(\mathrm{I} - p^{-s}\mathrm{Frob}_p|_{V_\ell(A)^{I_p}})^{-1}, \quad \Re(s) \gg 0.$$

It generalizes the Hasse–Weil L-function of E as $a_p(E)$ is the trace of the geometric Frobenius element Frob_p on $V_\ell(E) = T_\ell(E) \otimes \mathbb{Q}_\ell$.

Consider a symplectic motive M over $\mathbb{Q}$ of odd weight d and rank $2n$ with a nondegenerate alternating pairing $M \times M \to \mathbb{Q}(-d)$. For example, one can take the d-th cohomology group of a projective smooth variety with odd degree d, then the cup product gives a nondegenerate alternating pairing on M. The motivic L-function is defined by an Euler product

$$L(M,s) = \prod_p L_p(M_\ell, s)$$

where at each prime p, $L_p(s, M_\ell)$ is the Artin L-function of the Galois representation restricted to the decomposition group at p on the ℓ-adic realization M_ℓ of M. The Hodge decomposition $H^{p,q}(M)$ of the Hodge realization of M gives $2n$ pairs of numbers (p_i, q_i) with multiplicities equal to $\dim H^{p_i,q_i}(M)$. Notice that since we assume M is pure of odd weight d, the number $p_i + q_i = d$ is odd and $p_i \neq q_i$ with exactly n of $p_i - q_i$'s which are positive, say $1 \leq i \leq n$. The L-factor at infinity is

$$L_\infty(M,s) = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s - q_i), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

For example, when $M = H^1(E)$, one gets that $L_\infty(M,s) = 2(2\pi)^{-s}\Gamma(s)$. In general, as discussed in [7], the completed L-function

$$\Lambda(M,s) = L_\infty(M,s)L(M,s), \quad \Re(s) \gg 0,$$

has a conjectural functional equation

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(\check{M}, 1-s),$$

where in the case M is symplectic pure of weight d, one has $\check{M} \cong M(d)$ and

$$\epsilon(M,s) = \epsilon(M) \cdot f(M)^{\frac{d+1}{2}-s}$$

with integer $f(M)$, the global conductor

$$\prod_p p^{a_p},$$

where a_p is the Artin conductor at p, and $\epsilon(M) = \pm 1$, the global root number

$$\epsilon_\infty \cdot \prod_p \epsilon_p \quad \text{with } \epsilon_\infty = \prod_{i=1}^{n} (-1)^{q_i}.$$

The Local Langlands conjecture has been known for symplectic and orthogonal cases by Arthur, and the correspondence is given for generic representations of orthogonal groups by lifting to $\mathrm{GL}(2n)$ in the work of Soudry and Jiang [11] by functorial lift. The global Langlands conjecture predicts that the corresponding automorphic representation $\pi(M)$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ of a symplectic motive M should be cuspidal, tempered and, by a conjecture of Shahidi [18], globally generic. On the other hand, the Galois representation on M_ℓ gives a Langlands parameter of an irreducible generic representation $\pi_\nu(M)$ under Jiang's construction when restricts to a decomposition group at ν. The automorphic representation $\pi = \pi(M)$ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ is a restricted tensor product $\pi = \otimes' \pi_\nu$ of local representations π_ν of $\mathrm{SO}_{2n+1}(\mathbb{Q}_\nu)$ such that π_∞ is algebraic. Under the local–global compatibility it is expected that $\pi(M)_\nu$ agrees with $\pi_\nu(M)$. In other words, we would like to show that the adelic representation $\otimes' \pi_\nu(M)$ is automorphic and globally generic. By assuming Shahidi's conjecture we focus on the generic representation in the conjecturally generic L-packet of $\mathrm{SO}_{2n+1}(\mathbb{Q}_\nu)$ for each ν. Such representations are unique when exist.

4. Split special odd orthogonal groups

We let G be the split special orthogonal group of degree $2n+1$ over $\mathbb{Q}$. Let V be a standard representation of G which is a $(2n+1)$-dimensional vector space over $\mathbb{Q}$ generated by basis $e_1, e_2, \ldots, e_n, v_0, f_n, \ldots, f_2, f_1$ with G-invariant symmetric bilinear form

$$\begin{gathered}(e_i, f_j) = \delta_{ij}, \quad (e_i, e_j) = (f_i, f_j) = 0, \quad 1 \le i, j \le n, \text{ and} \\ (e_i, v_0) = (f_i, v_0) = 0, \quad 1 \le i \le n, \quad (v_0, v_0) = 2.\end{gathered}$$

Then V is a quadratic space and X generated by $e_1, e_2, \ldots, e_n$ is a maximal isotropic subspace whose dual $X^\vee$ is generated by $f_1, f_2, \ldots, f_n$. The vector v_0 is anisotropic and $\langle v_0 \rangle^\perp = X + X^\vee$. We write

$$V = V^+ + V^-$$

where V^+ is the positive definite subspace of $V \otimes \mathbb{R}$ generated by the $n+1$ vectors $e_1 + f_1, e_2 + f_2, \ldots, e_n + f_n$ and v_0, and V^- is the negative definite subspace of $V \otimes \mathbb{R}$ generated by the n vectors $e_1 - f_1, e_2 - f_2, \ldots, e_n - f_n$.

The p-adic group of G is $\mathrm{SO}(V \otimes \mathbb{Q}_p)$ with $G(\mathbb{Q}_p) = \mathrm{SO}_{2n+1}(\mathbb{Q}_p)$ and the real group of G is $\mathrm{SO}(V^+ + V^-)$ with $G(\mathbb{R}) = \mathrm{SO}_{n+1,n}(\mathbb{R})$. We fix a maximal compact subgroup K_∞ of $G(\mathbb{R})$ to be the subgroup $\mathrm{S}(\mathrm{O}(V^+) \times \mathrm{O}(V^-))$. There is an anti-isometry which gives an embedding $V^- \hookrightarrow V^+$ sending $e_i - f_i$ to $e_i + f_i$. It induces an embedding $\mathrm{O}(V^-) \hookrightarrow \mathrm{O}(V^+)$ and hence defines a diagonal embedding $\triangle : \mathrm{O}(V^-) \hookrightarrow K_\infty$. Let H_∞ be the subgroup of K_∞ which is the image of $\triangle$.

5. Real place

Given a symplectic pure motive M of odd weight d and rank $2n$ as in Section 2. Let (p_1, q_1), (p_2, q_2), ..., (p_n, q_n) be the n Hodge numbers with $p_i - q_i$ positive. The complexification of the Betti realization determines a $(2n)$-dimensional representation

$$\Phi = \sum_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(\frac{z}{\bar{z}}\right)^{\mu_i}, \quad \mu_i = \frac{p_i - q_i}{2}$$

when restricted to the Weil group $W_\mathbb{R}$. Arrange i such that the half integers μ_i satisfy

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 1/2.$$

The irreducible generic representation π_∞ of $G(\mathbb{R}) = \mathrm{SO}_{n+1,n}(\mathbb{R})$ with Harish-Chandra parameter

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n)$$

has Langlands parameter Φ and lies in the (limit) discrete series. One has $L_\infty(M, s) = L(\pi_\infty, s)$. Write K for $K_\infty = \mathrm{S}(\mathrm{O}_{n+1}(\mathbb{R}) \times \mathrm{O}_n(\mathbb{R}))$. We let T_K be the maximal torus of K. The group $G(\mathbb{R})$ has n non-compact simple roots

$$\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n$$

which we fix from now on and μ is in the closure of the positive Weyl chamber. The simple roots of $\mathrm{SO}_{n+1}(\mathbb{R}) \hookrightarrow K$ are the roots

$$\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_5, \ldots \text{ and } \epsilon_{n-1} - \epsilon_n \text{ if } n \text{ is even (or } \epsilon_n \text{ if } n \text{ is odd).}$$

The simple roots of $\mathrm{SO}_n(\mathbb{R}) \hookrightarrow K$ are the roots

$$\epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_6, \ldots \text{ and } \epsilon_n \text{ if } n \text{ is even (or } \epsilon_{n-1} - \epsilon_n \text{ if } n \text{ is odd).}$$

We write $\rho = \rho_c + \rho_n$ for the sum of positive roots, where $\rho = \left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}\right)$ is the sum of positive roots and $\rho_n = \left(\frac{n}{2}, \frac{n-1}{2}, \ldots, \frac{1}{2}\right)$ is the sum of positive non-compact roots and $\rho_c = \left(\frac{n-1}{2}, \frac{n-2}{2}, \ldots, 0\right)$ is the sum of positive compact roots.

The minimal K-type $\mathcal{V}$ of π_∞ has integral highest weight

$$\lambda = \mu + \rho_n - \rho_c = \left(\mu_1 + \frac{1}{2}, \mu_2 + \frac{1}{2}, \ldots, \mu_n + \frac{1}{2}\right)$$

for T_K and occurs with multiplicity one in the restriction of π_∞ to K (Schmid [17]). The classical branching law of compact orthogonal groups implies that in an irreducible representation of K_∞ of highest weight λ the trivial representation of H_∞ occurs with multiplicity 1 if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 1.$$

Indeed, the representation $\mathcal{V}$ is an induced representation by $\mathcal{V}_{\lambda^+} \otimes \mathcal{V}_{\lambda^-}$ from $\mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$ to K_∞ where $\mathcal{V}_{\lambda^+}$ (resp. $\mathcal{V}_{\lambda^-}$) denotes the irreducible representation of highest weight λ^+ (resp. λ^-) and the weights $\lambda^+ = (\lambda_1, \lambda_3, \ldots)$ and $\lambda^- = (\lambda_2, \lambda_4, \ldots)$ are interlaced. We are led to the following result which was first observed by Gross [8].

Theorem 5.1 *(Gross). The representation π_∞ has a unique line fixed by H_∞ in the minimal K-type of π_∞. This is true for all generic irreducible (limit) discrete series representations of* $\mathrm{SO}_{n+1,n}(\mathbb{R})$.

6. Finite places

The ring of adeles of $\mathbb{Q}$ decomposes to $\mathbb{A} = \mathbb{R} \times \hat{\mathbb{Q}}$ with $\hat{\mathbb{Q}} = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ the ring of finite adeles of $\mathbb{Q}$. The group G is a connected semisimple algebraic group over $\mathbb{Q}$ and K_∞ is the maximal compact subgroup of $G(\mathbb{R})$ determined up to conjugacy. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$, $\mathfrak{g}_\mathbb{C}$ be its complexification and $Z(\mathfrak{g}_\mathbb{C})$ is the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Fix a minimal K_∞-type $\mathcal{V}_\lambda$ of π_∞ of weight λ and an infinitesimal character χ of π_∞. Given any open compact subgroup $K = \prod_p' K_p$ of $G(\hat{\mathbb{Q}})$ the double coset space $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ is finite. We consider $\mathcal{A}_0(K, \mathcal{V}_\lambda, \chi)$, the space of cusp forms on $G(\mathbb{A})$ which are K-invariant, right translation by K_∞ via $\mathcal{V}_\lambda$ and annihilated by kernel of χ under action of $Z(\mathfrak{g}_\infty)$. These are called cusp forms of type $(K, \mathcal{V}_\lambda, \chi)$. Harish-Chandra [9] showed that $\mathcal{A}_0(K, \mathcal{V}_\lambda, \chi)$ is finite dimensional. By strong approximation of the spin groups, each cusp form in $\mathcal{A}_0(K, \mathcal{V}_\lambda, \chi)$ is determined by its restriction to the real group

$$\phi_\infty : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$$

where $\Gamma = G(\mathbb{R})K \cap G(\mathbb{Q})$ is an arithmetic subgroup of $G(\mathbb{Q})$. We denote this restriction on $G(\mathbb{R})$ by $\mathcal{A}_0(\Gamma, \mathcal{V}_\lambda, \chi)$.

When $n = 1$, the group G is GL_2 quotient by the center and we are in the case of the classical modular forms. Take Γ to be the congruence subgroup $\Gamma_0(N)$ and $K_\infty = \mathrm{SO}_2(\mathbb{R})$. The highest weight λ is a number (k) and $Z(\mathfrak{g}_\infty)$ is generated by the Casimir operator C. The holomorphic condition on the cusp forms implies that C acts by a scalar $\mu = -\frac{k}{2}\left(\frac{k}{2} - 1\right)$ which determines an infinitesimal character χ of $Z(\mathfrak{g}_\infty)$. The space $\mathcal{A}_0(\Gamma_0(N), \mathcal{V}_\lambda, \chi)$ is canonically isomorphic to the space of classical holomorphic cusp forms $S_k(\Gamma_0(N))$. In general to specify a specific line on the infinite place, like the holomorphic condition gives in $n = 1$, we make use of the existence of the line in $\mathcal{V}_\lambda$

fixed by the compact subgroup H_∞ shown in Theorem 5.1. This line determines the infinitesimal character χ.

Remark 6.1. In the case when $n = 1$, each cusp form in $\mathcal{A}_0(\Gamma_0(N), \mathcal{V}_{(k)}, \chi)$ fixed by H_∞ becomes a harmonic form with Fourier expansion $\sum_{n=1}^{\infty} a_n(e^{2\pi n i\tau} + e^{-2\pi n i\tau})$. This is because H_∞ contains $\left(\begin{smallmatrix} \pm1 & \\ & \pm1 \end{smallmatrix}\right)$.

We define the space of cusp form of *weight* λ and *level* Γ as cusp forms in $\mathcal{A}_0(\Gamma, \mathcal{V}_\lambda) := \hat{\oplus}_\chi \mathcal{A}_0(\Gamma, \mathcal{V}_\lambda, \chi)$ which are right H_∞-invariant. We shall denote the space of such cusp forms by $\mathcal{S}_\lambda(\Gamma)$.

The local theory of the newforms for $\mathrm{PGL}_2(\hat{\mathbb{Q}}) \simeq \mathrm{SO}_3(\hat{\mathbb{Q}})$ has been established by Casselman [3] with K taken to be the subgroup

$$K_0(N) = \left\{ g_f \in \mathrm{GL}_2(\hat{\mathbb{Z}}) \mid g_f \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N\hat{\mathbb{Z}} \right\} / \hat{\mathbb{Z}}^\times \mathrm{I}_2$$

such that Γ is the congruence subgroup $\Gamma_0(N)$. It is known that each elliptic curve E over $\mathbb{Q}$ associates with a newform ϕ_∞ of level the conductor of E. Write $K = \prod_p' K_p$. Casselman [3] showed that if π_p is an irreducible generic representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$, then there is a unique line in π_p fixed by K_p. Let ϕ be the corresponding cusp form on $\mathrm{PGL}_2(\mathbb{A})$ whose $\mathrm{PGL}_2(\hat{\mathbb{Q}})$-translate gives an abstract representation $\pi_f = \otimes_p' \pi_p$ of $G(\hat{\mathbb{Q}})$. Then Casselman's theory implies that ϕ spans the distinguished line π_f^K. We would like to generalize the local newform theory to rank $n \geq 2$ and define an open compact subgroup $K = \prod_p K_p$ of $G(\hat{\mathbb{Q}})$ such that for any irreducible generic representation π_p of $G(\mathbb{Q}_p)$, the subgroup K_p fixes a unique line in π_p.

We introduce a family of lattices in the standard representation V of G which are quadratic lattices and define a family of open compact subgroups of $G(\mathbb{Q}_p)$ at each finite place p in the next section. This construction was suggested by A. Brumer based on the case in GSp_4 where they suggest such family, called the paramodular groups, provides a correspondence from certain rational abelian surfaces of conductor N to the Siegel paramodular forms of weight 2 and level N. (See [2].)

7. The open compact subgroup of $G(\hat{\mathbb{Q}})$

Definition 7.1. Let $L(N)$ be the lattice in the quadratic space V defined as

$$L(N) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i + N\mathbb{Z}v_0 + \bigoplus_{i=1}^{n} N\mathbb{Z}f_i.$$

The symmetric bilinear form $N^{-1}(\ ,\ )$ restricted to $L(N)$ defines an integral bilinear form on $L(N)$. The special orthogonal group $\mathrm{SO}(L(N))$ is a group scheme over $\mathbb{Z}$.

Fix a prime p. Consider the group scheme $G_{p^m} = \mathrm{SO}(\mathbb{L}_{p^m})$ over $\mathbb{Z}_p$ with m the p-adic valuation of N, and $\mathbb{L}_{p^m} = L(N) \otimes \mathbb{Z}_p$. There is a decomposition [5] $\mathbb{L}_{p^m} = \mathbb{L}_{p^m}^{(0)} + \mathbb{L}_{p^m}^{(m)}$ for $m \geq 1$ where

$$\mathbb{L}_{p^m}^{(0)} = \bigoplus_{i=1}^{n} (\mathbb{Z}_p e_i \oplus p^m \mathbb{Z}_p f_i) \quad \text{and} \quad \mathbb{L}_{p^m}^{(m)} = p^m \mathbb{Z} v_0$$

into direct sum of quadratic lattices with respect to the integral quadratic form.

We define the open compact subgroup $K(p^m)$ as follows.

Definition 7.2. Let $J(p^m) = G_{p^m}(\mathbb{Z}_p)$ be the subgroup of $G(\mathbb{Q}_p)$ as the stabilizer of $\mathbb{L}_{p^m}$. Define the open compact subgroup $K(p^m)$ of $G(\mathbb{Q}_p)$ to be $J(1)$ if $m = 0$ and the kernel of the composition map

$$J(p^m) \xrightarrow{\text{mod } p} \mathrm{S}(\mathrm{O}_{2n}(\mathbb{F}_p) \times \mathrm{O}(\mathbb{L}_{p^m}^{(m)}/p\mathbb{L}_{p^m}^{(m)})) \to \mathrm{O}_{2n}(\mathbb{F}_p) \xrightarrow{\det} \{\pm 1\}$$

if $m \geq 1$, which is a normal subgroup of $J(p^m)$ of index 2.

Remark 7.3. These open compact subgroups $K(p^m)$ are conjugate to the congruence subgroups $K_0(N)$ of $\mathrm{PGL}_2(\mathbb{Q}_p) \simeq \mathrm{SO}_3(\mathbb{Q}_p)$ in the case when $n = 1$ and conjugate to the paramodular subgroups of $\mathrm{PGSp}_4(\mathbb{Q}_p) \simeq \mathrm{SO}_5(\mathbb{Q}_p)$ considered in [16] by Roberts and Schmidt in the case when $n = 2$.

Remark 7.4. The construction of the open compact subgroups $K(p^m)$ was first suggested by A. Brumer who pointed out the construction should be related to the orthogonal group of a quadratic lattice whose Gram matrix has $1, \dots, 1, 2p^m, 1, \dots, 1$ on the non-principal diagonal and zero elsewhere.

Consider H the subgroup of G defined as $\mathrm{SO}(\langle v_0 \rangle^{\perp})$ embedded as the stabilizer of v_0, which is isomorphic to the split even orthogonal group SO_{2n}. The open compact subgroups $K(p^m)$ of $G(\mathbb{Q}_p)$ enjoy the following properties.

Proposition 7.5. *(See [19].) (i) $K(1)$ is a hyperspecial maximal open compact subgroup.*

(ii) $K(p)$ is a maximal parahoric subgroup with reductive quotient $\mathrm{SO}_{2n}(\mathbb{F}_p)$.

(iii) The conjugacy classes of $K(p^m)$ form two descending chains by parity. More precisely, there exists an open compact subgroup $K_0(p^m)$ conjugate to $K(p^m)$ such that

$$K(1) = K_0(1) \supset K_0(p^2) \supset K_0(p^4) \supset \cdots \supset H_0 = \bigcap_{m:even} K_0(p^m),$$
$$K(p) = K_0(p) \supset K_0(p^3) \supset K_0(p^5) \supset \cdots \supset H_1 = \bigcap_{m:odd} K_0(p^m).$$

(iv) The subgroups H_0 and H_1 are hyperspecial open compact subgroups of $H(\mathbb{Q}_p)$.

Embed the subgroup $H(\mathbb{Q}_p)$ diagonally to $G(\mathbb{Q}_p) \times H(\mathbb{Q}_p)$. Gelbart and Piatetski-Shapiro [6] constructed a linear form Φ in $\mathrm{Hom}_{H(\mathbb{Q}_p)}(\pi \otimes \rho_s, \mathbb{C})$ nonzero for almost all $s \in \mathbb{C}$ with π any irreducible generic representation of $G(\mathbb{Q}_p)$ and ρ a generic unramified representation of $H(\mathbb{Q}_p)$. Since Φ is $H(\mathbb{Q}_p)$-invariant and any irreducible unramified representation is generated by any spherical vector, one sees Φ is uniquely determined by its values on $\pi^{H_i} \otimes f_s^i$ where f_s^i is the spherical vector fixed by H_i in ρ_s for $i = 0, 1$. The non-vanishing of Φ implies the H_i-fixed subspace π^{H_i} is nonzero. As one has

$$\pi^{H_0} = \bigcup_{m:even} \pi^{K_0(p^m)} \quad \text{and} \quad \pi^{H_1} = \bigcup_{m:odd} \pi^{K_0(p^m)}$$

it leads to the following result on existence.

Proposition 7.6. *(See [19].) Assume π is an irreducible generic representation of $G(\mathbb{Q}_p)$. For some integer $m \geq 0$ in both parity, the space $\pi^{K(p^m)}$ is nonzero.*

Remark 7.7. By using Bernstein–Zelevinsky's derivative theory and the result of Moy–Prasad [13] on Jacquet functors, when an irreducible representation is non-generic and supercuspidal the H_i-fixed subspaces are zero for $i = 0, 1$. Henceforth one has $\pi^{K(p^m)} = 0$ for all $m \geq 0$. The open compact subgroup $K(p^m)$ works specifically for generic representations.

Conjecture 7.8. *Assume π is an irreducible generic representation of $G(\mathbb{Q}_p)$ with conductor $a(\pi)$, equivalently the local ε-factor for the standard L-function of π has the form $\varepsilon(\pi)q^{-a(\pi)(s-\frac{1}{2})}$. Then there is a unique line in π fixed by $K(p^{a(\pi)})$.*

Moreover, further investigation on the functional equation given by the multiplicity one of the linear form Φ suggests that the number $\varepsilon(\pi) = \pm 1$ is determined by the action of the 2-group $J(p^{a(\pi)})/K(p^{a(\pi)})$ on the line $\pi^{K(p^{a(\pi)})}$. The element of order 2 which represents the non-trivial coset in $J(p^{a(\pi)})/K(p^{a(\pi)})$ generalizes the Atkin–Lehner elements in $\mathrm{PGL}_2(\hat{\mathbb{Q}})$ and $\mathrm{PGSp}_4(\hat{\mathbb{Q}})$.

When π is unramified, the conductor $a(\pi) = 0$ and the standard L-factor of π has degree $2n$. There exists a spherical vector and $\pi^{K(1)}$ is of dimension one. The linear form $\Phi(- \otimes f_s^0)$ takes value the L-factor on this line. To the other extreme, when π is supercuspidal, the L-factor $L(\pi, s) = 1$ and the conjecture has been known by the author's result.

Theorem 7.9. *(See [19].) Conjecture 7.8 holds when π is an irreducible generic supercuspidal representation.*

It is also not hard to see that the Steinberg representations are generic of conductor $2n - 1$ and the conjecture holds in this case. In general, by Muić's classification [14] on generic representations of $\mathrm{SO}_{2n+1}(\mathbb{Q}_p)$, every irreducible non-supercuspidal generic representation can be realized as the unique irreducible generic subquotient of a representation parabolically induced from a generic representation of a smaller standard Levi subgroup. While the theory for GL_n has been established by Jacquet, Piatetski-Shapiro and Shalika (see [10]), one can thus study generic non-supercuspidal representations by reducing to those of odd orthogonal groups of smaller ranks.

Remark 7.10. One expects the linear form Φ takes the value $L(\pi \otimes \rho, s)$ on the line $\pi^{K(p^{a(\pi)})}$, up to a local factor of Shahidi. In particular, the Whittaker functional of π does not vanish on the line.

The results of this section remain valid when one replaces $\mathbb{Q}_p$ (resp. $\mathbb{Z}_p$, p) with any non-archimedean local field (resp. the ring of integers, the maximal ideal).

8. Newforms

Assume $\pi = \otimes'_\nu \pi_\nu$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with π_∞ a (limit) discrete series representation of $G(\mathbb{R})$ whose minimal K_∞-type has highest weight $\mathcal{V}_\lambda$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $\mathcal{A}_0(\pi)$ be an irreducible summand of the space of cuspidal automorphic forms $\mathcal{A}_0(G)$ on G which is isomorphic to π and let $\mathcal{A}_0(\pi)_\lambda$ be the $\mathcal{V}_\lambda$-isotypic component of $\mathcal{A}_0(\pi)$ under action of K_∞. We assume Conjecture 7.8.

The space of newforms of π is defined as

$$\mathcal{A}_0(\pi)^{new} = \left\{ \phi \in \mathcal{A}_0(\pi)_\lambda \mid \phi(gk) = \phi(g),\ \forall\, k \in H_\infty \prod_p K(p^{a(\pi_p)}) \right\}$$

where $a(\pi_p)$ is the Artin conductor of the local component π_p. The multiplicity one of the minimal K_∞-type and Theorem 5.1, together with Conjecture 7.8 implies the space $\mathcal{A}_0(\pi)^{new}$ has dimensional one and spanned by a cusp form $\phi(\pi)$ with $\phi(\pi)_\infty \in \mathcal{S}_\lambda(\Gamma(N))$ where $\Gamma(N) = G(\mathbb{R})K(N) \cap G(\mathbb{Q})$ and $N = \prod_p p^{a(\pi_p)}$.

We would like to define a corresponding notion of "new" in the space of cusp forms $\mathcal{S}_\lambda(\Gamma(N))$ as in the Atkin–Lehner–Li theory for GL(2) as well as in the Paramodular Conjecture for GSp(4) and considered by Roberts and Schmidt [15]. As cusp forms in $\mathcal{S}_\lambda(\Gamma(N))$ are functions on $\Gamma(N)\backslash G(\mathbb{R})/H_\infty$ there is an action of $G(\hat{\mathbb{Q}})$ by strong approximation of the spin groups. Consider the action of the Hecke algebra, $\mathcal{H}(G(\hat{\mathbb{Q}})//K(N))$, the convolution algebra of locally constant, compactly supported bi-$K(N)$-invariant functions. It acts on both of the spaces $\mathcal{A}(\pi)^{new}$ and $\mathcal{S}_\lambda(\Gamma(N))$. The newform $\phi(\pi)$ and hence $\phi(\pi)_\infty$ are then simultaneous eigenvectors of the full algebra $\mathcal{H}(G(\hat{\mathbb{Q}})//K(N))$.

Now let $H(N)$ be the open compact subgroup $H(\hat{\mathbb{Q}}) \cap K(N)$ of $H(\hat{\mathbb{Q}})$. In [19], the author introduced an action of the commutative algebra $\mathcal{H}(H(\hat{\mathbb{Q}})//H(N))$ on $\mathcal{A}(\pi)^{H(N)}$ which can be carried to the space $\oplus_{n \geq N} \mathcal{A}_0(K(n), \mathcal{V}_\lambda, \chi)$ and hence $\oplus_{n \geq N} \mathcal{S}_\lambda(n)$. Such an action generalizes the (even) level raising operators for GL(2) and GSp(4). In [19], the author also defines two more operators θ_0, θ_1 that raise level by 1 generalizing those for GSp(4). These operators commute with each other. The author conjectured in [19] that cusp forms in $\mathcal{A}(\pi)$ fixed by $K(n)$, for some $n \geq N$, are obtained by iteratively applying these operators to $\phi(\pi)$ and the so-obtained cusp forms are called *oldforms*.

We follow the definition as in Atkin–Lehner–Li and define the space of newforms $\mathcal{S}_\lambda(\Gamma(N))^{new}$ in $\mathcal{S}_\lambda(\Gamma(N))$ to be the orthogonal complement of the subspace in $\mathcal{S}_\lambda(\Gamma(N))$ generated by the *oldforms*, forms obtained by iteratively applying θ_0, θ_1 and operators in $\mathcal{H}(H(\hat{\mathbb{Q}})//H(N))$ to a cusp form of smaller level. For example, for PGL_2 the operators in $\mathcal{H}(H(\hat{\mathbb{Q}})//H(N))$ consist of linear combinations of the operators $\phi_\infty(\tau) \mapsto \phi_\infty(d\tau)$, $d \in \mathbb{Z}$, which raise level by the LCM of d's. We note that under this definition it is clear that

$$\mathcal{A}_0(\pi)^{new} \subset \mathcal{S}_\lambda(\Gamma(N))^{new}.$$

We make the following conjecture. Let DS_λ denote the (limit) discrete series representation of $G(\mathbb{R})$ whose minimal K_∞-type has highest weight λ.

Conjecture 8.1.

$$\mathcal{S}_\lambda(\Gamma(N))^{new} = \bigoplus_{\substack{\pi \\ \pi_\infty = DS_\lambda \\ \pi \text{ has conductor } N}} \mathcal{A}_0(\pi)^{new}.$$

As a consequence of Conjecture 8.1, the Hecke operators preserve the space of newforms $\mathcal{S}_\lambda(\Gamma(N))^{new}$ and the space of oldforms $\mathcal{S}_\lambda(\Gamma(N))^{old}$, and the space of newforms $\mathcal{S}_\lambda(\Gamma(N))^{new}$ decomposes into direct sum of simultaneous eigenspaces of $\mathcal{H}(G(\hat{\mathbb{Q}})//K(N))$. This also implies that π occurs in $\mathcal{A}(G)$ with multiplicity one.

Combining this with the global Langlands conjecture, one obtains a precise refinement which associates each symplectic pure motive of rank $2n$ with Hodge weight λ and conductor N to a newform in $\mathcal{S}_\lambda(\Gamma(N))$. Furthermore, by Remark 7.10 we expect the nonzero Whittaker–Fourier coefficients of a globally generic cuspidal automorphic representation π do not vanish on the line of associated newforms and allow us to single out a normalized newform by the linear functional.

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