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On *L*-factors attached to generic representations of unramified U(2, 1)

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Abstract Let *G* be the unramified unitary group in three variables defined over a *p*-adic field with $p \neq 2$. In this paper, we establish a theory of newforms for the Rankin–Selberg integral for *G* introduced by Gelbart and Piatetski-Shapiro. We describe *L* and ε-factors defined through zeta integrals in terms of newforms. We show that zeta integrals of newforms for generic representations attain *L*-factors. As a corollary, we get an explicit formula for ε-factors of generic representations.

1 Introduction

This paper is the sequel to the author's works [10–12] on newforms for unramified U(2, 1). First of all, we review the theory of newforms for GL(2) by Casselman and Deligne. Let *F* be a non-archimedean local field of characteristic zero with ring of integers $\mathfrak{o}_F$ and its maximal ideal $\mathfrak{p}_F$. For each non-negative integer *n*, we define an open compact subgroup $\Gamma_0(\mathfrak{p}_F^n)$ of $\mathrm{GL}_2(F)$ by

$$\Gamma_0(\mathfrak{p}_F^n) = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F^n & 1 + \mathfrak{p}_F^n \end{pmatrix}^{\times}.$$

For an irreducible generic representation (π, V) of $\mathrm{GL}_2(F)$, we denote by $V(n)$ the $\Gamma_0(\mathfrak{p}_F^n)$-fixed subspace of *V*, that is,

$$V(n) = \{v \in V \mid \pi(k)v = v, \ k \in \Gamma_0(\mathfrak{p}_F^n)\}.$$

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Let U denote the unipotent radical of the upper-triangular Borel subgroup of $\mathrm{GL}_2(F)$. We regard a non-trivial additive character ψ_F of F with conductor $\mathfrak{o}_F$ as a character of U in the usual way, and denote by $\mathcal{W}(\pi, \psi_F)$ the Whittaker model of π with respected to ψ_F. Then the following theorem holds:

Theorem 1.1 [3] *Let (π, V) be an irreducible generic representation of $\mathrm{GL}_2(F)$.*

  (i) *There exists a non-negative integer n such that $V(n) \neq \{0\}$.*
 (ii) *Put $c(\pi) = \min\{n \geq 0 \mid V(n) \neq \{0\}\}$. Then the space $V(c(\pi))$ is one-dimensional.*
(iii) *For any $n \geq c(\pi)$, we have $\dim V(n) = n - c(\pi) + 1$.*
 (iv) *If v is a non-zero element in $V(c(\pi))$, then the corresponding Whittaker function W_v in $\mathcal{W}(\pi, \psi_F)$ satisfies $W_v(e) \neq 0$, where e denotes the identity element in $\mathrm{GL}_2(F)$.*

We call the integer $c(\pi)$ *the conductor* of π and $V(c(\pi))$ *the space of newforms* for π. Newforms and conductors relate to L and ε-factors as follows:

Theorem 1.2 [3,5] *Let π be an irreducible generic representation of $\mathrm{GL}_2(F)$.*

  (i) *Suppose that W is the newform in the Whittaker model of π. Then the corresponding Jacquet–Langlands's zeta integral $Z(s, W)$ attains the L-factor of π.*
 (ii) *The ε-factor $\varepsilon(s, \pi, \psi_F)$ of π is a constant multiple of $q_F^{-c(\pi)s}$, where q_F stands for the cardinality of the residue field of F.*

Similar results were obtained by Jacquet et al. [8] and Reeder [14] for $\mathrm{GL}(n)$. Recently, Roberts and Schmidt [15] established a theory of newforms for the irreducible representations of $\mathrm{GSp}(4)$ with trivial central characters. Our main concern is to establish a newform theory for unramified $\mathrm{U}(2, 1)$.

We review results in [10–12] comparing Theorems 1.1 and 1.2. Let $\mathrm{U}(2, 1)$ denote the unitary group in three variables associated to the unramified quadratic extension E/F. We assume that the residual characteristic of F is odd. In [12], the author introduced a family of open compact subgroups of $\mathrm{U}(2, 1)$, and defined the notion of conductors and newforms for generic representations. He proved an analog of Theorem 1.1 (i) and (ii) for all the generic representations, and that of (iii) and (iv) for the generic supercuspidal representations. For $\mathrm{U}(2, 1)$, we consider L and ε-factors defined through the Rankin–Selberg integral introduced by Gelbart and Piatetski-Shapiro [7] and Baruch [1]. In [11], the author showed a theorem analogous to Theorem 1.2 (ii) assuming Conjecture 4.1 in [11] on L-factors, which is an analog of Theorem 1.2 (i). In loc. cit., he also proved that his conjecture holds for the generic supercuspidal representations. To show the validity of his conjecture for the generic representations, he determined conductors of the generic non-supercuspidal representations, and gave an explicit realization of those newforms in [10]. In loc. cit., he also proved an analog of Theorem 1.1 (iii) and (iv) for the generic non-supercuspidal representations. Now we are ready to show that Conjecture 4.1 in [11] holds for all the generic representations of $\mathrm{U}(2, 1)$, that is, zeta integrals of newforms attain L-factors.

We explain our method. Unlike the cases of $\mathrm{GL}(n)$ and $\mathrm{GSp}(4)$, Gelbart and Piatetski-Shapiro's zeta integral involves a section which has the form $f(s, h, \Phi)$, where h is an element in $\mathrm{U}(1, 1)$ and Φ is a Schwartz function on F^2. Thus, the usual investigation on Whittaker functions is not enough to determine the L-factor, which is defined as the greatest common divisor of zeta integrals, and we can not use any explicit formula of L-factors for $\mathrm{U}(2, 1)$. However it is easy to determine the L-factors for $\mathrm{U}(2, 1)$ up to a multiple of $L_E(s, \mathbf{1})$ (Proposition 4.2). Here $L_E(s, \mathbf{1})$ stands for the Hecke-Tate factor of the trivial representation $\mathbf{1}$ of $E^\times$, and the section $f(s, h, \Phi)$ yields $L_E(s, \mathbf{1})$. We will compare zeta integral of newforms with our rough estimation of L-factors, and show that the difference is at most $L_E(s, \mathbf{1})$

(Lemma 3.5). Hence we can use the same trick in [11]. If the difference is $L_E(s, \mathbf{1})$, then it contradicts the fact that the ε-factor is monomial (see the proof of Theorem 3.6). So we conclude that zeta integrals of newforms attain *L*-factors.

The main body of this article is the proof of Lemma 3.5. For representations of conductor zero, we can use Casselman–Shalika's formula for the spherical Whittaker functions in [4]. To compute zeta integrals of newforms in positive conductor case, we follow the method by Roberts and Schmidt for GSp(4) in [15]. They utilized Hecke operators acting on the space of newforms, and obtained a formula for zeta integrals in terms of Hecke eigenvalues. There are two problems to apply their method to U(2, 1). Firstly, they assumed that representations of GSp(4) have trivial central characters. This assumption is essential in their computation of Hecke operators. Secondly, for an irreducible generic representation π of U(2, 1) whose conductor is positive, it will turns out that the degree of the *L*-factor of π is at most 4 with respect to q_F^{-s} (see Proposition 7.1 for example). Therefore we need two Hecke eigenvalues to describe zeta integrals of newforms. But, in the usual way, we have only one good Hecke operator which is represented by the element $\mathrm{diag}(\varpi, 1, \varpi^{-1})$, where ϖ is a uniformizer of *F*. We explain how to overcome these two problems. Let *V* denote the space of π, $V(n)$ its subspace consisting of the vectors fixed by the level *n* subgroup, and N_π the conductor of π. We consider the following two operators:

(1) The Hecke operator *T* on $V(N_\pi + 1)$ which is represented by the element diag $(\varpi, 1, \varpi^{-1})$;
(2) The composite map of the level raising operator $\theta' : V(N_\pi) \to V(N_\pi + 1)$ and the level lowering one $\delta : V(N_\pi + 1) \to V(N_\pi)$.

In [10], we have seen that both $V(N_\pi)$ and $V(N_\pi + 1)$ are one-dimensional, and hence the operators *T* and $\delta \circ \theta'$ have eigenvalues ν and λ. Since the central character of π is trivial on the level N_π subgroup, we can apply the method by Roberts and Schmidt to compute the Hecke operator *T* on $V(N_\pi + 1)$, and get a formula of zeta integrals of newforms in terms of ν and λ (Theorem 5.10).

We summarize the contents of this paper. In Sect. 2, we fix the notation for representations of unramified U(2, 1), and recall the theory of Rankin–Selberg integrals introduced by Gelbart, Piatetski-Shapiro and Baruch. In Sect. 3, we recall the notion of newforms for U(2, 1), and prove our main Theorem 3.6 assuming Lemma 3.5. In Sect. 4, we roughly estimate *L*-factors according to the classification of the irreducible representations of U(2, 1). In Sect. 5, we give a formula for zeta integrals of newforms in terms of two eigenvalues ν and λ. The proof of Lemma 3.5 is finished in Sect. 6. In Sect. 7, we give an example of an explicit computation of *L*-factors, for some non-supercuspidal representations. In Sect. 8, we determine *L*-factors of the depth zero supercuspidal representations.

A further direction of this research is to compare *L* and ε-factors defined by Gelbart and Piatetski-Shapiro's integral with those of *L*-parameters. It is also an interesting problem to generalize our result to other *p*-adic groups, for example, ramified U(2, 1) and unitary groups in odd variables.

2 Gelbart and Piatetski-Shapiro's integral

In Sect. 2.1, we fix our notation for the unramified group U(2, 1) that we use throughout this paper. In Sect. 2.2, we recall from [1] the theory of zeta integrals for U(2, 1) which is introduced by Gelbart and Piatetski-Shapiro in [7]. We also recall the definition of *L* and ε-factors attached to generic representations of U(2, 1) in Sects. 2.3 and 2.4 respectively.

2.1 Notations

Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}_F$ its ring of integers, $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$, and $\varpi = \varpi_F$ a uniformizer of F. We denote by $|\cdot|_F$ the absolute value of F normalized so that $|\varpi_F|_F = q^{-1}$, where $q = q_F$ is the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$. We use the analogous notation for any non-archimedean local fields. Throughout this paper, we assume that the residual characteristic of F is different from two.

Let $E = F[\sqrt{\epsilon}]$ be the unramified quadratic extension over F, where ϵ is a non-square element in $\mathfrak{o}_F^\times$. Then $\varpi = \varpi_F$ is a common uniformizer of E and F. Because the cardinality of the residue field of E is equal to q^2, we denote by $|\cdot|_E$ the absolute value of E normalized so that $|\varpi|_E = q^{-2}$. We realize the unramified unitary group in three variables defined over F as $G = \{g \in \mathrm{GL}_3(E) \mid {}^t\bar{g}Jg = J\}$, where $^-$ is the non-trivial element in $\mathrm{Gal}(E/F)$ and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by e the identity element of G.

Let B be the Borel subgroup of G consisting of the upper triangular elements in G, T its diagonal subgroup, and U the unipotent radical of B. We write $\hat{U}$ for the opposite of U. Then we have

$$U = \left\{ u(x, y) = \begin{pmatrix} 1 & x & y\sqrt{\epsilon} - x\bar{x}/2 \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x \in E, y \in F \right\}$$

$$= \left\{ \mathbf{u}(x, y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in E,\ y + \bar{y} + x\bar{x} = 0 \right\}$$

and

$$\hat{U} = \left\{ \hat{u}(x, y) = {}^t u(x, y) \mid x \in E, y \in F \right\}$$
$$= \left\{ \hat{\mathbf{u}}(x, y) = {}^t\mathbf{u}(x, y) \mid x, y \in E,\ y + \bar{y} + x\bar{x} = 0 \right\},$$

where ^{t} denotes the transposition of matrices. In most part of this paper, we write $u(x, y)$ for elements in U. The notion $\mathbf{u}(x, y)$ will appear only in the proofs of Lemmas 6.10 and 7.3. We identify the subgroup

$$H = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G \right\}$$

of G with $\mathrm{U}(1, 1)$. We set $B_H = B \cap H$, $U_H = U \cap H$ and $T_H = T \cap H$. Then B_H is the upper triangular Borel subgroup of H with Levi decomposition $B_H = T_H U_H$. There exists an isomorphism between $E^\times$ and T_H which is given by

$$t: E^\times \simeq T_H; a \mapsto t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix}.$$

A non-trivial additive character ψ_E of E defines the following character of U, which is also denoted by ψ_E:

$$\psi_E(u(x, y)) = \psi_E(x), \text{ for } u(x, y) \in U.$$

We say that a smooth representation π of G is *generic* if $\mathrm{Hom}_U(\pi, \psi_E) \neq \{0\}$. Let (π, V) be an irreducible generic representation of G. Then there exists a unique embedding of π into $\mathrm{Ind}_U^G \psi_E$ up to scalars. The image $\mathcal{W}(\pi, \psi_E)$ of π in $\mathrm{Ind}_U^G \psi_E$ is called *the Whittaker model of π*. For an element v in V, we denote by W_v the function in $\mathcal{W}(\pi, \psi_E)$ corresponding to v.

We identify the center Z of G with the norm-one subgroup E^1 of $E^\times$, and define open compact subgroups of Z by

$$Z_0 = Z, \ Z_n = Z \cap (1 + \mathfrak{p}_E^n), \ \text{for } n \geq 1.$$

For an irreducible admissible representation π of G, we define *the conductor n_π of* the central character ω_π of π by

$$n_\pi = \min\{n \geq 0 \,|\, \omega_\pi|_{Z_n} = 1\}.$$

2.2 Zeta integrals

Let $\mathcal{C}_c^\infty(F^2)$ denote the space of locally constant, compactly supported functions on F^2. For $\Phi \in \mathcal{C}_c^\infty(F^2)$ and $h \in H$, we define a function $f(s, h, \Phi)$ on $\mathbf{C}$ as in [11, section 3.1]. Let π be an irreducible generic representation of G. For $W \in \mathcal{W}(\pi, \psi_E)$ and $\Phi \in \mathcal{C}_c^\infty(F^2)$, we define the zeta integral $Z(s, W, \Phi)$ by

$$Z(s, W, \Phi) = \int_{U_H \backslash H} W(h) f(s, h, \Phi) dh,$$

where dh is the Haar measure on $U_H \backslash H$ normalized so that the volume of $U_H \backslash U_H(H \cap \mathrm{GL}_2(\mathfrak{o}_F))$ is one. By [1, Proposition 3.4], $Z(s, W, \Phi)$ absolutely converges to a function in $\mathbf{C}(q^{-2s})$ when $\mathrm{Re}(s)$ is sufficiently large.

Remark 2.1 Originally, Gelbart and Piatetski-Shapiro introduces a family of zeta integral of the form $Z(s, W, \Phi, \chi)$, where χ is a quasi-character of $E^\times$ (see [1]). In this paper, we consider the case when χ is a trivial character of $E^\times$.

2.3 L-factors

The L-factor of an irreducible generic representation π of G is defined as follows. Let I_π be the subspace of $\mathbf{C}(q^{-2s})$ spanned by $Z(s, W, \Phi)$ where $\Phi \in \mathcal{C}_c^\infty(F^2)$, $W \in \mathcal{W}(\pi, \psi_E)$ and ψ_E runs over all of the non-trivial additive characters of E. By [1, p. 331], I_π is a fractional ideal of $\mathbf{C}[q^{-2s}, q^{2s}]$ which contains $\mathbf{C}$. Thus, there exists a polynomial $P(X)$ in $\mathbf{C}[X]$ such that $P(0) = 1$ and $1/P(q^{-2s})$ generates I_π as $\mathbf{C}[q^{-2s}, q^{2s}]$-module. We define the L-factor $L(s, \pi)$ of π by

$$L(s, \pi) = \frac{1}{P(q^{-2s})}.$$

2.4 ε-Factors

Let ψ_F be a non-trivial additive character of F with conductor $\mathfrak{p}_F^{c(\psi_F)}$. We normalize the Haar measure on F^2 so that the volume of $\mathfrak{o}_F \oplus \mathfrak{o}_F$ equals to $q^{c(\psi_F)}$. For each $\Phi \in \mathcal{C}_c^\infty(F^2)$, let $\hat{\Phi}$ denote the Fourier transform of Φ defined in [1, section 2]. Then we have $\hat{\hat{\Phi}} = \Phi$ for all $\Phi \in \mathcal{C}_c^\infty(F^2)$. Due to [1, Corollary 4.8], there exists a rational function $\gamma(s, \pi, \psi_F, \psi_E)$ in q^{-2s} which satisfies

$$\gamma(s, \pi, \psi_F, \psi_E) Z(s, W, \Phi) = Z(1 - s, W, \hat{\Phi}).$$

We define the ε-factor $\varepsilon(s, \pi, \psi_F, \psi_E)$ of π by

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \gamma(s, \pi, \psi_F, \psi_E) \frac{L(s, \pi)}{L(1 - s, \widetilde{\pi})},$$

where $\widetilde{\pi}$ denotes the representation contragradient to π. By [11, Proposition 3.13], we have $L(s, \widetilde{\pi}) = L(s, \pi)$, and hence

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \gamma(s, \pi, \psi_F, \psi_E) \frac{L(s, \pi)}{L(1 - s, \pi)}. \tag{2.2}$$

For ε-factors, the following holds:

Proposition 2.3 [11, Proposition 3.15] *The ε-factor $\varepsilon(s, \pi, \psi_F, \psi_E)$ is a monomial in q^{-2s} which has the form*

$$\varepsilon(s, \pi, \psi_F, \psi_E) = \pm q^{-2n(s-1/2)},$$

for some $n \in \mathbf{Z}$.

3 Newforms and *L*-factors

In Sect. 3.1, we recall from [12] the notion of conductors and newforms for generic representations π of G. In Sect. 3.2, we prove our two main theorems assuming Lemma 3.5. We show that a newform for π attains the L-factor of π through Gelbart and Piatetski-Shapiro's integral (Theorem 3.6 (i)). Moreover we obtain the coincidence of the conductor of π and the exponent of q^{-2s} of the ε-factor of π (Theorem 3.6 (ii)). Lemma 3.5 will be proved in Sect. 6.

3.1 Newforms

For a non-negative integer n, we define an open compact subgroup K_n of G by

$$K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G.$$

For an irreducible generic representation (π, V) of G, we set

$$V(n) = \{v \in V \mid \pi(k)v = v, \ k \in K_n\}, \ n \geq 0.$$

We say that an element v in V is of level n if v lies in $V(n)$. By [12, Theorem 2.8], there exists a non-negative integer n such that $V(n)$ is not zero.

Definition 3.1 Let (π, V) be an irreducible generic representation of G. We call the integer $N_\pi = \min\{n \geq 0 \mid V(n) \neq \{0\}\}$ *the conductor of π* and elements in $V(N_\pi)$ *newforms for π.*

In [10], we gave an explicit formula for $\dim V(n)$, $n \geq N_\pi$. In particular, the following holds.

Theorem 3.2 [10, Corollary 5.2] *For any irreducible generic representation (π, V) of G, we have*

$$\dim V(N_\pi) = \dim V(N_\pi + 1) = 1.$$

Remark 3.3 Suppose that (π, V) is an irreducible generic representation of G. Then Z_{N_π} acts on $V(N_\pi)$ trivially. This implies $N_\pi \geq n_\pi$. The relation between N_π and n_π is crucial for the computation of zeta integrals of newforms in [11].

It follows from [12, Theorem 5.6] that the space $V(N_\pi)$ is one-dimensional. We shall relate newforms with Gelbart and Piatetski-Shapiro's integral. For $W \in \mathcal{W}(\pi, \psi_E)$, we define the zeta integral $Z(s, W)$ of W by

$$Z(s, W) = \int_{E^\times} W(t(a))|a|_E^{s-1} d^\times a.$$

Here we normalize the Haar measure $d^\times a$ on $E^\times$ so that the volume of $\mathfrak{o}_E^\times$ is one. One can show that the integral $Z(s, W)$ absolutely converges to a function in $\mathbf{C}(q^{-2s})$ when $\mathrm{Re}(s)$ is enough large, along the lines of the theory of zeta integrals for $\mathrm{GL}(2)$ by using [1, Proposition 3.3].

For each integer n, let Φ_n be the characteristic function of $\mathfrak{p}_F^n \oplus \mathfrak{o}_F$. We denote by $L_E(s, \chi)$ the L-factor of a quasi-character χ of $E^\times$, that is,

$$L_E(s, \chi) = \begin{cases} \dfrac{1}{1 - \chi(\varpi)q^{-2s}}, & \text{if } \chi \text{ is unramified;} \\ 1, & \text{if } \chi \text{ is ramified.} \end{cases}$$

We write $\mathbf{1}$ for the trivial character of $E^\times$. One important property of our compact subgroups $\{K_n\}_{n \geq 0}$ is that $K_n \cap H$ is a maximal compact subgroup of H for any $n \geq 0$. So we obtain an Iwasawa decomposition $H = U_H T_H (K_n \cap H)$. By using this decomposition, we get the following:

Proposition 3.4 [11, Proposition 3.5] *Let n be any non-negative integer. Suppose that a function W in $\mathcal{W}(\pi, \psi_E)$ is fixed by K_n. Then we have*

$$Z(s, W, \Phi_n) = Z(s, W)L_E(s, \mathbf{1}).$$

If the conductor of ψ_E is $\mathfrak{o}_E$, then it follows from [10, Proposition 5.1] that any non-zero element $v \in V(N_\pi)$ satisfies $W_v(e) \neq 0$. Due to Theorem 3.2, there exists a unique newform v for π such that $W_v(e) = 1$. We state the key lemma which will be proved in Sect. 6.

Lemma 3.5 *Suppose that the conductor of ψ_E is $\mathfrak{o}_E$. Let v be the element in $V(N_\pi)$ which satisfies $W_v(e) = 1$. Then we have*

$$Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi) \text{ or } L(s, \pi)/L_E(s, \mathbf{1}).$$

In the following theorem, we will show that the latter ($Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)/L_E(s, \mathbf{1})$) is never the case, that is, the zeta integral of the newform attains the L-factor.

3.2 The main theorem

We shall prove our main theorem, which is an analog of Theorem 1.1.

Theorem 3.6 *We fix an additive character ψ_E of E with conductor $\mathfrak{o}_E$. For any irreducible generic representation π of G, we have the followings:*

(i) *Let v be the element in $V(N_\pi)$ such that $W_v(e) = 1$. Then we have*

$$Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi).$$

(ii) *If ψ_F has conductor $\mathfrak{o}_F$, then we have*

$$\varepsilon(s, \pi, \psi_F, \psi_E) = q^{-2N_\pi(s-1/2)}.$$

Proof By Lemma 3.5, we have $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)$ or $L(s, \pi)/L_E(s, \mathbf{1})$. Suppose that $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)/L_E(s, \mathbf{1})$. Take an additive character ψ_F of F whose conductor is $\mathfrak{o}_F$. Then, by [11, Proposition 3.9], we get

$$Z(1-s, W_v, \hat{\Phi}_{N_\pi}) = q^{-2N_\pi(s-1/2)} Z(1-s, W_v, \Phi_{N_\pi}),$$

and hence

$$Z(1-s, W_v, \hat{\Phi}_{N_\pi}) = q^{-2N_\pi(s-1/2)} L(1-s, \pi)/L_E(1-s, \mathbf{1})$$

by assumption. Due to (2.2), we obtain

$$\frac{Z(1-s, W_v, \hat{\Phi}_{N_\pi})}{L(1-s, \pi)} = \varepsilon(s, \pi, \psi_F, \psi_E) \frac{Z(s, W_v, \Phi_{N_\pi})}{L(s, \pi)},$$

so that

$$q^{-2N_\pi(s-1/2)} \frac{1}{L_E(1-s, \mathbf{1})} = \varepsilon(s, \pi, \psi_F, \psi_E) \frac{1}{L_E(s, \mathbf{1})}.$$

This implies that $\varepsilon(s, \pi, \psi_F, \psi_E)$ is not a monomial in q^{-2s}, which contradicts Proposition 2.3. Therefore we conclude that $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)$. This implies (i). Now the assertion (ii) follows from [11, Theorem 4.3]. □

4 An estimation of *L*-factors

The remaining of this paper is devoted to the proof of Lemma 3.5. In this section, we roughly estimate the L-factors of generic representations of G. To state our result, we fix the notation for parabolically induced representations. For a quasi-character μ_1 of $E^\times$ and a character μ_2 of E^1, we define a quasi-character $\mu = \mu_1 \otimes \mu_2$ of T by

$$\mu \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} = \mu_1(a)\mu_2(b), \quad \text{for } a \in E^\times \text{ and } b \in E^1.$$

We regard μ as a quasi-character of B which is trivial on U. Let $\mathrm{Ind}_B^G(\mu)$ denote the normalized parabolic induction. Then the space of $\mathrm{Ind}_B^G(\mu)$ is that of locally constant functions $f : G \to \mathbf{C}$ which satisfy

$$f(bg) = \delta_B(b)^{1/2} \mu(b) f(g), \quad \text{for } b \in B, \ g \in G,$$

where δ_B is the modulus character of B. Note that

$$\delta_B \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} = |a|_E^2, \quad \text{for } a \in E^\times \text{ and } b \in E^1.$$

The group G acts on the space of $\mathrm{Ind}_B^G(\mu)$ by the right translation.

Let (π, V) be an irreducible generic representation of G. To study the integral $Z(s, W)$ of $W \in \mathcal{W}(\pi, \psi_E)$, we recall from [12, section 4.2] some properties of the restriction of

Whittaker functions to T_H. Let W be a function in $\mathcal{W}(\pi, \psi_E)$. Under the identification $T_H \simeq E^\times$, the restriction $W|_{T_H}$ of W to T_H is a locally constant function on $E^\times$, and there exists an integer n such that supp $W|_{T_H} \subset \mathfrak{p}_E^n$. We set $V(U) = \langle \pi(u)v - v \mid v \in V, \ u \in U \rangle$. Then for any element v in $V(U)$, the function $W_v|_{T_H}$ lies in $\mathcal{C}_c^\infty(E^\times)$.

The next lemma follows along the lines in the theory of zeta integrals for GL(2). However we give a proof for the reader's convenience. In the below, we denote by $\overline{\mu}_1$ the quasi-character of $E^\times$ defined by $\overline{\mu}_1(a) = \mu_1(\overline{a})$, $a \in E^\times$.

Lemma 4.1 *Let π be an irreducible generic representation of G and W a function in $\mathcal{W}(\pi, \psi_E)$.*

(i) *Suppose that π is supercuspidal. Then $Z(s, W)$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$.*
(ii) *Suppose that π is a proper submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, for some μ_1 and μ_2. Then $Z(s, W)$ belongs to $L_E(s, \mu_1)\mathbf{C}[q^{-2s}, q^{2s}]$.*
(iii) *Suppose that $\pi = \operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, for some μ_1 and μ_2. Then the integral $Z(s, W)$ lies in $L_E(s, \mu_1)L_E(s, \overline{\mu}_1^{-1})\mathbf{C}[q^{-2s}, q^{2s}]$.*

Proof Let $V_U = V/V(U)$ be the normalized Jacquet module of π. The group T acts on V_U by $\delta_B^{-1/2}\pi$.

(i) If π is supercuspidal, then we have $V_U = \{0\}$. Since W is associated to an element in $V = V(U)$, the function $W|_{T_H}$ lies in $\mathcal{C}_c^\infty(E^\times)$, and hence $Z(s, W)$ belongs to $\mathbf{C}[q^{-2s}, q^{2s}]$.

(ii) In this case, V_U is isomorphic to $\mu_1 \otimes \mu_2$ as T-module. Take $v \in V$ such that $W = W_v$. If v lies in $V(U)$, then by the proof of (i), $\mathbf{C}[q^{-2s}, q^{2s}]$ contains $Z(s, W)$, so does $L_E(s, \mu_1)\mathbf{C}[q^{-2s}, q^{2s}]$. Suppose that v does not belong to $V(U)$. Since V_U is isomorphic to μ_1 as T_H-module, we see that the element $\delta_B^{-1/2}(t(a))\pi(t(a))v - \mu_1(a)v$ lies in $V(U)$ for any $a \in E^\times$. Set $v' = \delta_B^{-1/2}(t(a))\pi(t(a))v - \mu_1(a)v$. One can observe that $Z(s, W_{v'}) = (|a|_E^{-s} - \mu_1(a))Z(s, W_v)$. So $(|a|_E^{-s} - \mu_1(a))Z(s, W_v)$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$ for all $a \in E^\times$.

Suppose that μ_1 is ramified. Then we can find $a \in \mathfrak{o}_E^\times$ such that $\mu_1(a) \neq 1$. Thus, we see that $(1 - \mu_1(a))Z(s, W_v)$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$. If μ_1 is unramified, then we have $(q^{2s} - \mu_1(\varpi))Z(s, W_v) \in \mathbf{C}[q^{-2s}, q^{2s}]$ by putting $a = \varpi$. These imply that $Z(s, W_v)$ lies in $L_E(s, \mu_1)\mathbf{C}[q^{-2s}, q^{2s}]$, as required.

(iii) In the case when $\pi = \operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, there is a T-submodule V_1 of V_U such that $V_U/V_1 \simeq \mu_1 \otimes \mu_2$ and $V_1 \simeq \overline{\mu}_1^{-1} \otimes \mu_2$. Then we can easily show the assertion by repeating the argument in the proof of (ii) twice. □

According to the classification of representations of G, we obtain the following estimation of L-factors:

Proposition 4.2 *Let π be an irreducible generic representation of G.*

(i) *Suppose that π is supercuspidal. Then $L_E(s, \mathbf{1})$ divides $L(s, \pi)$, that is, $L(s, \pi)L_E(s, \mathbf{1})^{-1}$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$.*
(ii) *Suppose that π is a proper submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, for some μ_1 and μ_2. Then $L_E(s, \mu_1)L_E(s, \mathbf{1})$ divides $L(s, \pi)$.*
(iii) *Suppose that $\pi = \operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, for some μ_1 and μ_2. Then $L(s, \pi)$ is divided by $L_E(s, \mu_1)L_E(s, \overline{\mu}_1^{-1})L_E(s, \mathbf{1})$.*

Proof Let W and Φ be functions in $\mathcal{W}(\pi, \psi_E)$ and $\mathcal{C}_c^\infty(F^2)$ respectively. Note that $W(h)$ and $f(s, h, \Phi)$ are right smooth functions on H. So the integral $Z(s, W, \Phi)$ can be written as a linear combination of $Z(s, W')f(s, e, \Phi')$, where $W' \in \mathcal{W}(\pi, \psi_E)$ and $\Phi' \in \mathcal{C}_c^\infty(F^2)$. By the theory of zeta integrals for GL(1), we see that $f(s, e, \Phi')$ lies in $L_E(s, \mathbf{1})\mathbf{C}[q^{-2s}, q^{2s}]$. So the assertion follows from Lemma 4.1. □

5 Zeta integrals of newforms

The proof of Lemma 3.5 will be done by comparing zeta integrals of newforms with Proposition 4.2. To this end, we give a formula for zeta integrals of newforms in this section. Let (π, V) be an irreducible generic representation of G. If N_π is zero, then Gelbart and Piatetski-Shapiro in [7] computed zeta integrals of newforms by using Casselman–Shalika's formula for the spherical Whittaker functions in [4]. So we treat only representations with $N_\pi > 0$ here. In this case, we will follow the method by Roberts and Schmidt for GSp(4). In [15, section 7.4], they give a formula for zeta integrals of newforms for representations of PGSp(4) with $N_\pi \geq 2$, in two Hecke eigenvalues. For U(2, 1), we need two eigenvalues to describe zeta integrals of newforms. But we have only one nice Hecke operator. The key in our computation is to consider the spaces $V(N_\pi)$ and $V(N_\pi + 1)$ simultaneously, which are both one-dimensional. In Sect. 5.1, we recall the definition of the level raising operator $\theta' : V(N_\pi) \to V(N_\pi + 1)$. The first eigenvalue ν is defined in Sect. 5.2 as that of the Hecke operator T on $V(N_\pi + 1)$. The second one λ is introduced in Sect. 5.3 as the eigenvalue of the composite map of θ' and the level lowering operator $\delta : V(N_\pi + 1) \to V(N_\pi)$. Recall that we need the condition $n > n_\pi$ to describe the level lowering operator δ on $V(n)$ (see [11, Lemma 5.9]). Since we always have $N_\pi + 1 > n_\pi$ by Remark 3.3, we can calculate the operator δ on $V(N_\pi + 1)$. In Sect. 5.4, we describe zeta integrals of newforms explicitly with ν and λ (Theorem 5.10).

5.1 The level raising operator θ'

From now on, we assume that the conductor of ψ_E is $\mathfrak{o}_E$. Let (π, V) be an irreducible generic representation of G whose conductor N_π is positive. We abbreviate $N = N_\pi$. Let θ' denote the level raising operator from $V(N)$ to $V(N + 1)$ defined in [12, section 3]. By [12, Proposition 3.3], we have

$$\theta' v = \pi(\zeta^{-1})v + \sum_{x \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{-N}} \pi(u(0, x))v, \quad v \in V(N), \tag{5.1}$$

where

$$\zeta = \begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{pmatrix}.$$

We fix a newform v in $V(N)$, and set

$$c_i = W_v(\zeta^i), \quad d_i = W_{\theta'v}(\zeta^i),$$

for $i \in \mathbf{Z}$.

Lemma 5.2 *For $i \in \mathbf{Z}$, we have $d_i = c_{i-1} + qc_i$.*

Proof By (5.1), we obtain

$$W_{\theta'v}(\zeta^i) = W_v(\zeta^{i-1}) + \sum_{x \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{-N}} W_v(\zeta^i u(0, x)),$$

for $i \in \mathbf{Z}$. Since $\zeta^i u(0, x) = u(0, \varpi^{2i}x)\zeta^i$ and $\psi_E(u(0, \varpi^{2i}x)) = 1$, we obtain $W_v(\zeta^i u(0, x)) = W_v(\zeta^i)$, and hence

$$W_{\theta' v}(\zeta^i) = W_v(\zeta^{i-1}) + q W_v(\zeta^i).$$

This implies the lemma.   □

5.2 The eigenvalue v

Let T denote the Hecke operator on $V(N + 1)$ defined in [11, subsection 5.1]. For $w \in V(N + 1)$, we have

$$Tw = \frac{1}{\mathrm{vol}(K_N)} \int_{K_N \zeta K_N} \pi(g) w \, dg = \sum_{k \in K_N / K_N \cap \zeta K_N \zeta^{-1}} \pi(k\zeta) w.$$

By Theorem 3.2, the space $V(N + 1)$ is one-dimensional. So there exists a complex number v, which is called *the Hecke eigenvalue* of T, such that

$$Tw = vw$$

for all $w \in V(N + 1)$. For $w \in V(N + 1)$, we set

$$w' = \sum_{y \in \mathfrak{p}_E^N / \mathfrak{p}_E^{N+1}} \sum_{z \in \mathfrak{p}_F^N / \mathfrak{p}_F^{N+1}} \pi(\hat{u}(y, z)) w. \tag{5.3}$$

For each $i \in \mathbf{Z}$, we put

$$d_i' = W_{(\theta' v)'}(\zeta^i).$$

Then we have the following

Lemma 5.4 *For $i \geq 0$, we have* $vd_i = d_{i-1}' + q^4 d_{i+1}$.

Proof By [11, Lemma 5.4], we obtain

$$v\theta' v = T\theta' v = \pi(\zeta^{-1})(\theta' v)' + \sum_{\substack{a \in \mathfrak{o}_E / \mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N} / \mathfrak{p}_F^{1-N}}} \pi(u(a, b)\zeta) \theta' v. \tag{5.5}$$

Thus, we get

$$v W_{\theta' v}(\zeta^i) = W_{(\theta' v)'}(\zeta^{i-1}) + \sum_{\substack{a \in \mathfrak{o}_E / \mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N} / \mathfrak{p}_F^{1-N}}} W_{\theta' v}(\zeta^i u(a, b)\zeta),$$

for $i \geq 0$. Note that $\zeta^i u(a, b) = u(\varpi^i a, \varpi^{2i} b)\zeta^i$ and $\psi_E(u(\varpi^i a, \varpi^{2i} b)) = \psi_E(\varpi^i a) = 1$ because $a \in \mathfrak{o}_E$ and ψ_E has conductor $\mathfrak{o}_E$. Hence we have $W_{\theta' v}(\zeta^i u(a, b)\zeta) = W_{\theta' v}(\zeta^{i+1})$, and hence

$$v W_{\theta' v}(\zeta^i) = W_{(\theta' v)'}(\zeta^{i-1}) + q^4 W_{\theta' v}(\zeta^{i+1}).$$

This completes the proof.   □

5.3 The eigenvalue λ

The central character ω_π of π is trivial on $Z_N = Z \cap K_N$. Since the group $Z_N K_{N+1}$ acts on $V(N + 1)$ trivially, we can define the level lowering operator $\delta : V(N + 1) \to V(N)$ by

$$\delta w = \frac{1}{\mathrm{vol}(K_N \cap (Z_N K_{N+1}))} \int_{K_N} \pi(k) w \, dk = \sum_{k \in K_N / K_N \cap (Z_N K_{N+1})} \pi(k) w,$$

for $w \in V(N+1)$. Theorem 3.2 implies that $V(N)$ is of dimension one. So there exists a complex number λ such that

$$\lambda v = \delta \theta' v$$

for all $v \in V(N)$.

Lemma 5.6 *We have*

$$d_i' + q^2 d_{i+1} = \lambda c_i, \ i \geq 0,$$
$$d_{-1}' = 0.$$

Proof Since N is positive and ω_π is trivial on Z_N, we have $N + 1 \geq 2$ and $N + 1 > n_\pi$. So we can apply [11, Lemma 5.9], and get

$$\lambda v = \delta \theta' v = (\theta' v)' + \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \pi(\zeta u(y, 0))\theta' v. \qquad (5.7)$$

Hence we obtain

$$\lambda W_v(\zeta^i) = W_{(\theta' v)'}(\zeta^i) + \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} W_{\theta' v}(\zeta^{i+1} u(y, 0)),$$

for $i \in \mathbf{Z}$. Because $\zeta^{i+1} u(y, 0) = u(\varpi^{i+1} y, 0)\zeta^{i+1}$ and $\psi_E(u(\varpi^{i+1} y, 0)) = \psi_E(\varpi^{i+1} y)$, we have $W_{\theta' v}(\zeta^{i+1} u(y, 0)) = \psi_E(\varpi^{i+1} y) W_{\theta' v}(\zeta^{i+1})$. So we get

$$\lambda W_v(\zeta^i) = W_{(\theta' v)'}(\zeta^i) + \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \psi_E(\varpi^{i+1} y) W_{\theta' v}(\zeta^{i+1}).$$

If $i \geq 0$, then we have $\psi_E(\varpi^{i+1} y) = 1$ because $\varpi^{i+1} y \in \mathfrak{o}_E$ and ψ_E has conductor $\mathfrak{o}_E$. So we have

$$\lambda W_v(\zeta^i) = W_{(\theta' v)'}(\zeta^i) + q^2 W_{\theta' v}(\zeta^{i+1}).$$

This implies $\lambda c_i = d_i' + q^2 d_{i+1}$, for $i \geq 0$.

If $i = -1$, then we have $\sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \psi_E(y) = 0$, and hence $\lambda W_v(\zeta^{-1}) = W_{(\theta' v)'}(\zeta^{-1})$. Due to [12, Corollary 4.6], we get $W_v(\zeta^{-1}) = 0$. So we obtain $W_{(\theta' v)'}(\zeta^{-1}) = 0$. This implies $d_{-1}' = 0$.   □

5.4 Zeta integrals of newforms in ν and λ

We get the following recursion formula for $c_i = W_v(\zeta^i), i \geq 0$.

Lemma 5.8 *We have*

$$(\nu + q^2 - \lambda)c_i + q(\nu + q^2 - q^3)c_{i+1} = q^5 c_{i+2}, \ i \geq 0,$$
$$(\nu - q^3)c_0 = q^4 c_1.$$

Proof The assertion follows from Lemmas 5.2, 5.4 and 5.6. For the second equation, we note that $c_{-1} = W_v(\zeta^{-1})$ is equal to zero because of [12, Corollary 4.6].   □

By Lemma 5.8, we get the following formula of zeta integrals of newforms.

Proposition 5.9 *Let (π, V) be an irreducible generic representation of G whose conductor N_π is positive. For any $v \in V(N_\pi)$, we have*

$$Z(s, W_v) = \frac{(1 - q^{-2s})W_v(e)}{1 - (v + q^2 - q^3)q^{-2}q^{-2s} - (v + q^2 - \lambda)q^{-1}q^{-4s}}.$$

Proof For $v \in V(N_\pi)$, it follows from [12, Corollary 4.6] that supp $W_v|_{T_H} \subset \mathfrak{o}_E$. Since $W_v|_{T_H}$ is $\mathfrak{o}_E^\times$-invariant, we obtain

$$Z(s, W_v) = \sum_{i=0}^{\infty} W_v(\zeta^i)|\varpi^i|_E^{s-1} = \sum_{i=0}^{\infty} c_i q^{2i(1-s)}.$$

Put $\alpha = (v + q^2 - q^3)q^{-4}$ and $\beta = (v + q^2 - \lambda)q^{-5}$. Then by Lemma 5.8, we have

$$c_{i+2} = \alpha c_{i+1} + \beta c_i, \ i \geq 0.$$

So we obtain

$$Z(s, W_v) = c_0 + c_1 q^{2-2s} + \sum_{i=0}^{\infty} (\alpha c_{i+1} + \beta c_i)q^{2(i+2)(1-s)}$$

$$= c_0 + c_1 q^{2-2s} + \beta q^{4-4s} \sum_{i=0}^{\infty} c_i q^{2i(1-s)} + \alpha q^{2-2s} \sum_{i=0}^{\infty} c_i q^{2i(1-s)} - \alpha c_0 q^{2-2s}$$

$$= c_0 + c_1 q^{2-2s} + \beta q^{4-4s} Z(s, W_v) + \alpha q^{2-2s} Z(s, W_v) - \alpha c_0 q^{2-2s}$$

$$= c_0 + (c_1 - \alpha c_0)q^{2-2s} + (\alpha q^{2-2s} + \beta q^{4-4s})Z(s, W_v).$$

Thus we have

$$Z(s, W_v) = \frac{c_0 + (c_1 - \alpha c_0)q^{2-2s}}{1 - \alpha q^{2-2s} - \beta q^{4-4s}}$$

$$= \frac{c_0(1 - q^{-2s})}{1 - (v + q^2 - q^3)q^{-2}q^{-2s} - (v + q^2 - \lambda)q^{-1}q^{-4s}}.$$

In the last equality, we use the equation $c_1 - \alpha c_0 = -q^{-2}c_0$ from Lemma 5.8. Now the proof is complete. $\square$

Theorem 5.10 *We assume that ψ_E has conductor $\mathfrak{o}_E$. Let (π, V) be an irreducible generic representation of G whose conductor N_π is positive. For the newform v in $V(N_\pi)$ which satisfies $W_v(e) = 1$, we have*

$$Z(s, W_v, \Phi_{N_\pi}) = \frac{1}{1 - (v + q^2 - q^3)q^{-2}q^{-2s} - (v + q^2 - \lambda)q^{-1}q^{-4s}},$$

where v is the eigenvalue of the Hecke operator T on $V(N_\pi + 1)$ and λ is that of the operator $\delta\theta'$ on $V(N_\pi)$.

Proof The theorem follows from Propositions 3.4 and 5.9. $\square$

6 Proof of Lemma 3.5

In this section, we prove Lemma 3.5. An irreducible generic representation π of G is either supercuspidal or a submodule of $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$, for some μ_1 and μ_2. We distinguish the cases according to the form of *L*-factors:

(I) π is an unramified principal series representation, that is, $\pi = \operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is unramified and μ_2 is trivial (Sect. 6.1);

(II) π is a supercuspidal representation (Sect. 6.2);

(III) π is a submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is ramified (Sect. 6.3);

(IV) π is a submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is unramified, but π is not an unramified principal series representation (Sect. 6.5).

Remark 6.1 We remark that representations in cases (II)–(IV) have positive conductors. If π is generic and supercuspidal, then by [12, Corollary 5.5], we have $N_\pi \geq 2$. Conductors of the non-supercuspidal representations are determined in [10]. By the proof of Proposition 5.1 in [10], if π is non-supercuspidal and generic, then $N_\pi = 0$ implies that π is an unramified principal series representation. In particular, the representations in case (IV) are just the irreducible generic subrepresentations of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$ with positive conductors, where μ_1 runs over the unramified quasi-characters of $E^\times$.

6.1 Proof of Lemma 3.5: Case (I)

Let μ_1 be an unramified quasi-character of $E^\times$ and μ_2 the trivial character of E^1. Suppose that $\pi = \operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$ is irreducible. We show that Lemma 3.5 holds for π. In this case, π has a non-zero K_0-fixed vector. This implies $N_\pi = 0$. Let V denote the space of π and let v be the element in $V(0)$ which satisfies $W_v(e) = 1$. By [7, (4.7)], we obtain

$$Z(s, W_v, \Phi_0) = L_E(s, \mu_1) L_E(s, \overline{\mu}_1^{-1}) L_E(s, \mathbf{1}).$$

because $\overline{\mu}_1 = \mu_1$. Due to Proposition 4.2 (iii), we have

$$Z(s, W_v, \Phi_0) = L(s, \pi) = L_E(s, \mu_1) L_E(s, \overline{\mu}_1^{-1}) L_E(s, \mathbf{1}), \tag{6.2}$$

which completes the proof of Lemma 3.5 in this case.

6.2 Proof of Lemma 3.5: Case (II)

Let (π, V) be an irreducible generic supercuspidal representation of G. We show the validity of Lemma 3.5 for π. In this case, we have $L(s, \pi) = 1$ or $L_E(s, \mathbf{1})$ by Proposition 4.2 (i). Let v be the element in $V(N_\pi)$ which satisfies $W_v(e) = 1$. Then it follows from Theorem 5.10 that $Z(s, W_v, \Phi_{N_\pi})$ has the form $1/P(q^{-2s})$, for some $P(X) \in \mathbf{C}[X]$. Note that $Z(s, W_v, \Phi_{N_\pi})/L(s, \pi)$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$ by the definition of $L(s, \pi)$. So one may observe that $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)$ or $L(s, \pi) L_E(s, \mathbf{1})^{-1}$, as required.

6.3 Proof of Lemma 3.5: Case (III)

Suppose that an irreducible generic representation (π, V) of G is a submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is a ramified quasi-character of $E^\times$. In this case, we have $L(s, \pi) = 1$ or $L_E(s, \mathbf{1})$ by Proposition 4.2 (ii) and (iii) because $L_E(s, \mu_1) = L_E(s, \overline{\mu}_1^{-1}) = 1$. Thus we can show that Lemma 3.5 is valid for π as in Sect. 6.2.

6.4 Eigenvalues ν and λ

To prove Lemma 3.5 for representations in case (IV), we need more information on the eigenvalues ν and λ defined in Sect. 5. Suppose that an irreducible generic representation (π, V) of G is a submodule of $\operatorname{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is an unramified quasi-character of $E^\times$. We assume that N_π is positive.

Remark 6.3 We identify the center Z of G with E^1. In the case when μ_1 is unramified, the representation $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$ admits the central character $\omega_\pi = \mu_2$, so does π. Since π has a non-zero K_{N_π}-fixed vector, $\omega_\pi = \mu_2$ is trivial on $Z_{N_\pi} = E^1 \cap (1 + \mathfrak{p}_E)^{N_\pi}$.

We may regard an element in V as a function in $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$. It follows from [10, Corollary 4.3] that every non-zero element f in $V(N_\pi)$ satisfies $f(e) \neq 0$. By using this property of newforms, we show a relation between ν and λ. We abbreviate $N = N_\pi$.

Lemma 6.4 *For $f \in V(N)$, we have*

$$(\theta' f)(e) = (q^2 \mu_1(\varpi)^{-1} + q) f(e).$$

In particular, $(\theta' f)(e) \neq 0$ for all non-zero $f \in V(N)$.

Proof By (5.1), we have

$$(\theta' f)(e) = f(\zeta^{-1}) + \sum_{x \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{-N}} f(u(0, x)).$$

Since f belongs to $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$, we obtain $f(\zeta^{-1}) = \delta_B^{1/2}(\zeta^{-1}) \mu_1(\varpi^{-1}) f(e) = q^2 \mu_1(\varpi)^{-1} f(e)$ and $f(u(0, x)) = f(e)$. So we have

$$(\theta' f)(e) = q^2 \mu_1(\varpi)^{-1} f(e) + q f(e) = (q^2 \mu_1(\varpi)^{-1} + q) f(e),$$

as required. For the second assertion, it suffices to claim that $q^2 \mu_1(\varpi)^{-1} + q \neq 0$. Since μ_1 is unramified, if $q^2 \mu_1(\varpi)^{-1} + q = 0$, then we have $\mu_1|_{F^\times} = \omega_{E/F} \cdot |\ |_F^{-1}$, where $\omega_{E/F}$ is the non-trivial character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$. If this is the case, then it follows from [9] that $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$ is reducible, and it contains no irreducible generic subrepresentations (see [10, Lemma 3.6] for instance). This contradicts the assumption that $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$ contains π. □

We obtain the following relation between ν and λ:

Lemma 6.5 *We have $\lambda = (\nu + q^2 - q^2 \mu_1(\varpi))(1 + q^{-1} \mu_1(\varpi))$.*

Proof For $f \in V(N)$, we put $(\theta' f)' = \sum_{y \in \mathfrak{p}_E^N/\mathfrak{p}_E^{N+1}} \sum_{z \in \mathfrak{p}_F^N/\mathfrak{p}_F^{N+1}} \pi(\hat{u}(y, z)) \theta' f$ as in (5.3). Then by (5.5), we obtain

$$\nu(\theta' f)(e) = (\theta' f)'(\zeta^{-1}) + \sum_{\substack{a \in \mathfrak{o}_E/\mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{1-N}}} (\theta' f)(u(a, b)\zeta).$$

Since we regard $\theta' f$ and $(\theta' f)'$ as functions in $\text{Ind}_B^G(\mu_1 \otimes \mu_2)$, we have

$$(\theta' f)'(\zeta^{-1}) = |\varpi|_E^{-1} \mu_1(\varpi^{-1})(\theta' f)'(e) = q^2 \mu_1(\varpi^{-1})(\theta' f)'(e)$$

and

$$(\theta' f)(u(a, b)\zeta) = |\varpi|_E \mu_1(\varpi)(\theta' f)(e) = q^{-2} \mu_1(\varpi)(\theta' f)(e).$$

So we get

$$\nu(\theta' f)(e) = q^2 \mu_1(\varpi^{-1})(\theta' f)'(e) + q^2 \mu_1(\varpi)(\theta' f)(e). \tag{6.6}$$

On the other hand, by (5.7), we obtain

$$\lambda f(e) = (\theta' f)'(e) + \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} (\theta' f)(\zeta u(y, 0)),$$

and get

$$\lambda f(e) = (\theta' f)'(e) + \mu_1(\varpi)(\theta' f)(e) \tag{6.7}$$

in a similar fashion. By (6.6) and (6.7), we have

$$\nu(\theta' f)(e) = q^2 \mu_1(\varpi^{-1})(\lambda f(e) - \mu_1(\varpi)(\theta' f)(e)) + q^2 \mu_1(\varpi)(\theta' f)(e).$$

According to Lemma 6.4, we obtain

$$(\nu + q^2 - q^2 \mu_1(\varpi))(q^2 \mu_1(\varpi)^{-1} + q) f(e) = q^2 \mu_1(\varpi^{-1})\lambda f(e).$$

If $f \in V(N)$ is not zero, then we get $f(e) \neq 0$. So this completes the proof. $\qquad \square$

By Lemma 6.5, we get a formula for zeta integrals of newforms with only ν.

Proposition 6.8 *We fix a non-trivial additive character ψ_E of E whose conductor is $\mathfrak{o}_E$. Let (π, V) be an irreducible generic representation of G whose conductor N_π is positive and v the newform for π such that $W_v(e) = 1$. Suppose that π is a subrepresentation of $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is an unramified quasi-character of $E^\times$. Then we have*

$$Z(s, W_v, \Phi_{N_\pi}) = L_E(s, \mu_1) \frac{1}{1 - (\nu + q^2 - q^3 - q^2 \mu_1(\varpi))q^{-2}q^{-2s}}.$$

Proof By Lemma 6.5, we get

$$\lambda - \nu - q^2 = (\nu + q^2 - q^3 - q^2 \mu_1(\varpi))q^{-1}\mu_1(\varpi),$$

and hence

$$1 - (\nu + q^2 - q^3)q^{-2}q^{-2s} - (\nu + q^2 - \lambda)q^{-1}q^{-4s}$$
$$= (1 - (\nu + q^2 - q^3 - q^2 \mu_1(\varpi))q^{-2}q^{-2s})(1 - \mu_1(\varpi)q^{-2s}).$$

So the assertion follows from Theorem 5.10. $\qquad \square$

We shall describe the Hecke eigenvalue ν by values of a function f in $V(N_\pi)$. Recall that ν is the eigenvalue of the Hecke operator T on $V(N_\pi + 1)$. We abbreviate $N = N_\pi$. One has

$$\nu g = Tg = \sum_{k \in K_{N+1}/K_{N+1} \cap \zeta K_{N+1}\zeta^{-1}} \pi(k\zeta)v,$$

for $g \in V(N+1)$. For any integer i, we set

$$\gamma_i = \hat{u}(\varpi^i, 0) = \begin{pmatrix} 1 & & \\ \varpi^i & 1 & \\ -\varpi^{2i}/2 & -\varpi^i & 1 \end{pmatrix} \quad \text{and} \quad t_i = \begin{pmatrix} & & \varpi^{-i} \\ & 1 & \\ \varpi^i & & \end{pmatrix}.$$

We note that if $n \geq 0$, then t_n lies in K_n. Recall that Lemma 5.2 in [11] gave a complete set of representatives for $K_{N+1}/K_{N+1} \cap \zeta K_{N+1}\zeta^{-1}$. Thus we obtain

$$\nu g = \sum_{\substack{y \in \mathfrak{o}_E/\mathfrak{p}_E \\ z \in \mathfrak{p}_F^{-N}/\mathfrak{p}_F^{1-N}}} \pi(t_{N+1}u(y, z)\zeta)g + \sum_{\substack{a \in \mathfrak{o}_E/\mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{1-N}}} \pi(u(a, b)\zeta)g.$$

Because $t_{N+1}u(y,z)\zeta t_{N+1} = \zeta^{-1}\hat{u}(-\varpi^N\overline{y}, \varpi^{2N}z)$, we get

$$vg = \pi(\zeta^{-1}) \sum_{\substack{y \in \mathfrak{p}_E^N/\mathfrak{p}_E^{N+1} \\ z \in \mathfrak{p}_F^N/\mathfrak{p}_F^{N+1}}} \pi(\hat{u}(y,z))g + \sum_{\substack{a \in \mathfrak{o}_E/\mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{1-N}}} \pi(u(a,b)\zeta)g. \qquad (6.9)$$

See Lemma 5.4 in [11] for details. The following lemma describes v by the values of a function g in $V(N_\pi + 1)$ at e and γ_{N_π}.

Lemma 6.10 *For $g \in V(N_\pi + 1)$, we have*

$$vg(e) = (q^2(\mu_1(\varpi) + \mu_1(\varpi)^{-1}) + q^3 - q^2)g(e) + q^2(q^2 - 1)\mu_1(\varpi)^{-1}g(\gamma),$$

where $\gamma = \gamma_{N_\pi}$.

Proof We abbreviate $N = N_\pi$. By (6.9), we have

$$vg(e) = \sum_{\substack{y \in \mathfrak{p}_E^N/\mathfrak{p}_E^{N+1} \\ z \in \mathfrak{p}_F^N/\mathfrak{p}_F^{N+1}}} g(\zeta^{-1}\hat{u}(y,z)) + \sum_{\substack{a \in \mathfrak{o}_E/\mathfrak{p}_E \\ b \in \mathfrak{p}_F^{-1-N}/\mathfrak{p}_F^{1-N}}} g(u(a,b)\zeta).$$

Since we regard g as an element in $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, we have

$$g(\zeta^{-1}\hat{u}(y,z)) = |\varpi|_E^{-1}\mu_1(\varpi)^{-1}g(\hat{u}(y,z)) = q^2\mu_1(\varpi)^{-1}g(\hat{u}(y,z))$$

and

$$g(u(a,b)\zeta) = g(\zeta) = |\varpi|_E\mu_1(\varpi)g(e) = q^{-2}\mu_1(\varpi)g(e).$$

Thus, we get

$$vg(e) = q^2\mu_1(\varpi)^{-1} \sum_{\substack{y \in \mathfrak{p}_E^N/\mathfrak{p}_E^{N+1} \\ z \in \mathfrak{p}_F^N/\mathfrak{p}_F^{N+1}}} g(\hat{u}(y,z)) + q^2\mu_1(\varpi)g(e).$$

To prove the assertion, it is enough to claim that (i) $g(\hat{u}(y,z)) = g(\gamma)$, for $y \notin \mathfrak{p}_E^{N+1}$, $z \in \mathfrak{p}_F^{N+1}$ and (ii) $g(\hat{u}(y,z)) = q^{-2}\mu_1(\varpi)g(e)$ for $z \notin \mathfrak{p}_F^{N+1}$. Actually, we obtain

$$vg(e) = q^2\mu_1(\varpi)^{-1}(g(e) + (q^2-1)g(\gamma) + q^{-2}\mu_1(\varpi)q^2(q-1)g(e)) + q^2\mu_1(\varpi)g(e)$$
$$= (q^2(\mu_1(\varpi) + \mu_1(\varpi)^{-1}) + q^3 - q^2)g(e) + q^2(q^2-1)\mu_1(\varpi)^{-1}g(\gamma),$$

as required.

We shall show the claim. (i) Suppose that $y \notin \mathfrak{p}_E^{N+1}$ and $z \in \mathfrak{p}_F^{N+1}$. Then there exists $a \in \mathfrak{o}_E^\times$ such that $t(a)\hat{u}(y,0)t(a)^{-1} = \hat{u}(\varpi^N, 0) = \gamma$. Since g is fixed by K_{N+1}, we have

$$g(\hat{u}(y,z)) = g(\hat{u}(y,0)) = g(t(a)^{-1}\gamma t(a)) = g(t(a)^{-1}\gamma).$$

Because we assume that μ_1 is unramified, we get $g(\hat{u}(y,z)) = \mu_1(a^{-1})g(\gamma) = g(\gamma)$.

(ii) If $z \notin \mathfrak{p}_F^{N+1}$, then the element $x = z\sqrt{\epsilon} - y\overline{y}/2$ lies in $\mathfrak{p}_E^N \backslash \mathfrak{p}_E^{N+1}$. Using the notation in Sect. 2.1, we write $\hat{u}(y,z) = \hat{\mathbf{u}}(y,x)$. Then we have

$$\hat{\mathbf{u}}(y,x) = \mathbf{u}(-\overline{y}/x, 1/x)\mathrm{diag}(\varpi^{N+1}/\overline{x}, -\overline{x}/x, \varpi^{-1-N}x)t_{N+1}\mathbf{u}(-\overline{y}/x, 1/x).$$

One can observe that $t_{N+1}\mathbf{u}(-\overline{y}/x, 1/x)$ lies in K_{N+1}. Since g is an element in $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ fixed by K_{N+1}, we have

$$g(\hat{u}(y,z)) = g(\hat{\mathbf{u}}(y,x)) = g(\mathrm{diag}(\varpi^{N+1}/\overline{x}, -\overline{x}/x, \varpi^{-1-N}x)).$$

The assumption $x \in \mathfrak{p}_E^N \backslash \mathfrak{p}_E^{N+1}$ implies $\varpi^{N+1}/\overline{x} \in \varpi \mathfrak{o}_E^\times$, so we get $g(\hat{u}(y,z)) = q^{-2}\mu_1(\varpi)\mu_2(-\overline{x}/x)g(e)$. Note that $x + \overline{x} + y\overline{y} = 0$, and hence $-\overline{x}/x = 1 + y\overline{y}/x$. Since $y \in \mathfrak{p}_E^N$ and $x \in \mathfrak{p}_E^N \backslash \mathfrak{p}_E^{N+1}$, we obtain $-\overline{x}/x \in 1 + \mathfrak{p}_E^N$. Thus, by Remark 6.3, we see that $\mu_2(-\overline{x}/x) = 1$, so that $g(\hat{u}(y,z)) = q^{-2}\mu_1(\varpi)g(e)$. $\qquad\square$

Applying Lemma 6.10 to $g = \theta' f$, where $f \in V(N_\pi)$, we get the following

Lemma 6.11 *For any non-zero element f in $V(N_\pi)$, we have*

$$\nu = q^2(\mu_1(\varpi) + \mu_1(\varpi)^{-1}) + q^3 - q^2$$
$$+ q^2(q^2 - 1)\mu_1(\varpi)^{-1}(q^2\mu_1(\varpi)^{-1} + q)^{-1}(\theta' f)(\gamma)/f(e),$$

where $\gamma = \gamma_{N_\pi}$.

Proof Put $g = \theta' f \in V(N_\pi + 1)$. By Lemma 6.4, we have $g(e) = (q^2\mu_1(\varpi)^{-1} + q)f(e) \neq 0$. So the assertion follows from Lemma 6.10. $\qquad\square$

We apply Lemma 6.11 to zeta integrals of newforms.

Proposition 6.12 *Under the same assumption of Proposition 6.8, we have*

$$Z(s, W_v, \Phi_{N_\pi}) = L_E(s, \mu_1)\frac{1}{1 - \alpha q^{-2s}}.$$

Here α is given by

$$\alpha = \mu_1(\varpi)^{-1} + \mu_1(\varpi)^{-1}(q^2 - 1)(q^2\mu_1(\varpi)^{-1} + q)^{-1}(\theta' f)(\gamma_{N_\pi})/f(e),$$

for any non-zero function f in $V(N_\pi)$.

Proof The proposition follows from Proposition 6.8 and Lemma 6.11. $\qquad\square$

6.5 Proof of Lemma 3.5: Case (IV)

We shall finish the proof of Lemma 3.5. The remaining representations are those in case (IV). Let (π, V) be an irreducible generic representation of G whose conductor is positive. We suppose that π is a subrepresentation of $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is unramified.

Firstly, we assume that π is a proper submodule of $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$. Then Proposition 4.2 (ii) implies that $L(s, \pi) = L_E(s, \mu_1)$ or $L_E(s, \mu_1)L_E(s, \mathbf{1})$. Let v be the newform in $V(N_\pi)$ such that $W_v(e) = 1$. It follows from Proposition 6.12 that $Z(s, W_v, \Phi_{N_\pi})$ has the form $L_E(s, \mu_1) \cdot (1/P(q^{-2s}))$, for some $P(X) \in \mathbf{C}[X]$. Because $Z(s, W_v, \Phi_{N_\pi})/L(s, \pi)$ lies in $\mathbf{C}[q^{-2s}, q^{2s}]$, we must have $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)$ or $L(s, \pi)/L_E(s, \mathbf{1})$.

Secondly, we consider the case when $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$. The assumption $N_\pi > 0$ implies that μ_2 is not trivial. In this case, we can show Lemma 3.5 by comparing Proposition 4.2 (iii) with the following one in a similar fashion:

Proposition 6.13 *Let μ_1 be an unramified quasi-character of $E^\times$ and μ_2 a non-trivial character of E^1. Suppose that $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ is irreducible. Then we have*

$$Z(s, W_v, \Phi_{N_\pi}) = L_E(s, \mu_1)L_E(s, \overline{\mu}_1^{-1}),$$

where v is the newform in $V(N_\pi)$ such that $W_v(e) = 1$.

Proof Set $\gamma = \gamma_{N_\pi}$. Since μ_1 is unramified, we have $\overline{\mu}_1 = \mu_1$. By Proposition 6.12, it enough to show that $\theta' f(\gamma) = 0$, for any functions f in $V(N_\pi)$. By [10, Theorem 2.4 (ii)], the space of $K_{N_\pi+1}$-fixed vectors in $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ is one-dimensional and consists of the functions whose supports are contained in $BK_{N_\pi+1}$ since we assume that μ_1 is unramified. Due to [10, Lemma 2.1], the sets $B\gamma K_{N_\pi+1}$ and $BK_{N_\pi+1} = B\gamma_{N_\pi+1}K_{N_\pi+1}$ are disjoint. So for any $f \in V(N_\pi)$, we get $(\theta' f)(\gamma) = 0$ because $\theta' f$ is fixed by $K_{N_\pi+1}$. This completes the proof. □

Now the proof of Lemma 3.5 is complete.

7 An example of a computation of *L*-factors

Let (π, V) be an irreducible generic representation of G whose conductor N_π is positive. Suppose that π is a subrepresentation of $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, where μ_1 is an unramified quasi-character of $E^\times$ and μ_2 is a character of E^1. In this section, we determine the *L*-factor of π by using the results in Sect. 6.4.

7.1 Irreducible case

Suppose that $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ is irreducible. Then we have $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ and μ_2 is not trivial because we assume that $N_\pi > 0$.

Proposition 7.1 *Let μ_1 be an unramified quasi-character of $E^\times$ and μ_2 a non-trivial character of E^1. Suppose that $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ is irreducible. Then we have*

$$L(s, \pi) = L_E(s, \mu_1)L_E(s, \overline{\mu}_1^{-1}).$$

Proof Theorem 3.6 and Proposition 6.13 imply the assertion. □

7.2 Reducible case

Suppose that $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ is reducible. Recall that we assume that $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ contains an irreducible generic subrepresentation π. So, by [9], there are the following three cases:

(RU1) $\mu_1 = |\cdot|_E$ and μ_2 is trivial: Then π is the Steinberg representation St_G of G and $N_\pi = 2$ by [10, Proposition 3.4 (i)]. (Proposition 7.6).
(RU2) $\mu_1|_{F^\times} = \omega_{E/F} \cdot |\cdot|_F$, where $\omega_{E/F}$ denotes the non-trivial character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$. By [10, Proposition 3.7 (i)], we have $N_\pi = c(\mu_2) + 1$. (Propositions 7.5 and 7.6).
(RU3) μ_1 is trivial and μ_2 is not trivial: Then due to [10, Proposition 3.8 (i)], we get $N_\pi = c(\mu_2)$. (Proposition 7.2).

Here $c(\mu_2)$ denotes the conductor of μ_2, that is,

$$c(\mu_2) = \min\{n \geq 0 \,|\, \mu_2|_{E^1 \cap (1+\mathfrak{p}_E)^n} = 1\}.$$

We fix a non-trivial additive character ψ_E of E with conductor $\mathfrak{o}_E$. Let v be the newform for π such that $W_v(e) = 1$. Then by Theorem 3.6, we have $Z(s, W_v, \Phi_{N_\pi}) = L(s, \pi)$. We regard elements in V as functions in $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$. By Proposition 6.12, to determine $L(s, \pi) = Z(s, W_v, \Phi_{N_\pi})$, it is enough to compute $(\theta' f)(\gamma_{N_\pi})/f(e)$, where f is a non-zero function in $V(N_\pi)$. We shall determine $(\theta' f)(\gamma_{N_\pi})/f(e)$ explicitly, for each case.

7.3 Case (RU3)

We consider the case (RU3).

Proposition 7.2 *Let μ_2 be a non-trivial character of E^1 and (π, V) the irreducible generic subrepresentation of $\mathrm{Ind}_B^G(\mathbf{1} \otimes \mu_2)$. Then we have*

$$L(s, \pi) = L_E(s, \mathbf{1})^2.$$

Proof It follows from [10, Proposition 3.8] that $V(n)$ coincides with the space of K_n-fixed vectors in $\mathrm{Ind}_B^G(\mathbf{1} \otimes \mu_2)$ for all n. So we may apply the argument in the proof of Proposition 6.13, and get $(\theta' f)(\gamma_{N_\pi}) = 0$, for any $f \in V(N_\pi)$. By Proposition 6.12, we obtain $Z(s, W_v, \Phi_{N_\pi}) = L_E(s, \mathbf{1})^2$, where v is the newform in $V(N_\pi)$ such that $W_v(e) = 1$. The assertion follows from this and Theorem 3.6. □

7.4 Case (RU2-I)

Let us consider the case (RU2). We further assume that μ_2 is trivial. The remaining case is treated in the next subsection. Then $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ has the trivial central character, so does π. By [10, Proposition 3.7 (i)], we get $N_\pi = 1$. Since $\mu_1|_{F^\times} = \omega_{E/F} \cdot |_F$, we have $\mu_1(\varpi) = -q^{-1}$.

Lemma 7.3 *For $f \in V(1)$, we have*

$$(\theta' f)(\gamma_1) = (q + 1)f(e).$$

Proof We abbreviate $\gamma = \gamma_1$. Set $g = \theta' f \in V(2)$ and $\gamma' = t_2 \gamma t_2 = u(-\varpi^{-1}, 0)$. We have $\gamma = t_2 \gamma' t_2 = \zeta^{-1} t_1 \gamma' t_2$. Since g is a function in $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ which is fixed by K_2 and $t_2 \in K_2$, we obtain $g(\gamma) = g(\zeta^{-1} t_1 \gamma' t_2) = q^2 \mu_1(\varpi^{-1}) g(t_1 \gamma')$. By (5.1), we get

$$g(t_1 \gamma') = f(t_1 \gamma' \zeta^{-1}) + \sum_{x \in \mathfrak{p}_F^{-2}/\mathfrak{p}_F^{-1}} f(t_1 \gamma' u(0, x)),$$

and hence

$$g(\gamma) = q^2 \mu_1(\varpi^{-1}) f(t_1 \gamma' \zeta^{-1}) + q^2 \mu_1(\varpi^{-1}) \sum_{x \in \mathfrak{p}_F^{-2}/\mathfrak{p}_F^{-1}} f(t_1 \gamma' u(0, x)). \qquad (7.4)$$

Firstly, we get $t_1 \gamma' \zeta^{-1} = t_1 \zeta^{-1} \zeta \gamma' \zeta^{-1}$. Note that $t_1 \zeta^{-1} = \zeta t_1$ and $\zeta \gamma' \zeta^{-1} = u(-1, 0)$. We get $t_1 \gamma' \zeta^{-1} = \zeta t_1 u(-1, 0)$. Since $t_1 u(-1, 0) \in K_1$ and $f \in V(1)$, we obtain

$$f(t_1 \gamma' \zeta^{-1}) = f(\zeta t_1 u(-1, 0)) = f(\zeta) = q^{-2} \mu_1(\varpi) f(e).$$

Secondly, we get $t_1 \gamma' u(0, x) = t_1 u(-\varpi^{-1}, x) = \hat{u}(1, \varpi^2 x) t_1$. Since $t_1 \in K_1$ and $f \in V(1)$, we obtain

$$f(t_1 \gamma' u(0, x)) = f(\hat{u}(1, \varpi^2 x) t_1) = f(\hat{u}(1, \varpi^2 x)).$$

Set $z = \varpi^2 x \sqrt{\epsilon} - 1/2$. Then z lies in $\mathfrak{o}_E^\times$ because $\varpi^2 x \in \mathfrak{p}_E^2$. With the notation in Sect. 2.1, we write $\hat{u}(1, \varpi^2 x) = \hat{u}(1, z)$. We use the relation

$$\hat{u}(1, z) = u(-1/\bar{z}, 1/z) \mathrm{diag}(\varpi/\bar{z}, -\bar{z}/z, \varpi^{-1} z) t_1 u(-1/z, 1/z).$$

By $z \in \mathfrak{o}_E^\times$, we have $t_1 \mathbf{u}(-1/z, 1/z) \in K_1$. Recall that f is a function in $(\mathrm{Ind}_B^G \mu_1 \otimes \mu_2)$ which is fixed by K_1. So we obtain

$$f(t_1 \gamma' u(0, x)) = f(\mathrm{diag}(\varpi/\overline{z}, -\overline{z}/z, \varpi^{-1}z)) = q^{-2}\mu_1(\varpi)f(e)$$

because z lies in $\mathfrak{o}_E^\times$ and we assume that μ_2 is trivial. Finally, by (7.4), we get $g(\gamma) = (q+1)f(e)$, as required.   □

Proposition 7.5 *Let μ_1 be an unramified quasi-character of $E^\times$ which satisfies $\mu_1|_{F^\times} = \omega_{E/F}|\cdot|_F$, and μ_2 the trivial character of E^1. For the irreducible generic subrepresentation π of $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, we have*

$$L(s, \pi) = L_E(s, \mu_1)L_E(s, \mathbf{1}).$$

Proof We may apply Proposition 6.12. Due to Lemma 7.3, the number α in Proposition 6.12 satisfies

$$\alpha = \mu_1(\varpi)^{-1} + \mu_1(\varpi)^{-1}(q^2 - 1)(q^2\mu_1(\varpi)^{-1} + q)^{-1}(q+1) = 1,$$

since $\mu_1(\varpi) = -q^{-1}$. Now the assertion follows from Theorem 3.6 and Proposition 6.12. □

7.5 Cases (RU1) and (RU2-II)

Suppose that an irreducible generic representation π of G is a subrepresentation of $\mathrm{Ind}_G(\mu_1 \otimes \mu_2)$. We assume that μ_1 and μ_2 satisfy one of the following conditions:

(1) $\mu_1 = |\cdot|_E$ and μ_2 is trivial;
(2) μ_1 is an unramified quasi-character of $E^\times$ such that $\mu_1|_{F^\times} = \omega_{E/F}|\cdot|_F$, and μ_2 is a non-trivial character of E^1.

In the first case, we have $N_\pi = 2$ by [10, Proposition 3.4] (i), and π has the trivial central character. In the second case, we get $N_\pi = c(\mu_2) + 1 \geq 2$ by [10, Proposition 3.7 (i)], and $n_\pi = c(\mu_2)$ by Remark 6.3.

Proposition 7.6 *Suppose that an irreducible generic representation π satisfies one of the assumptions in this subsection. Then we have*

$$L(s, \pi) = L_E(s, \mu_1).$$

Proof In both cases, we have $N_\pi \geq 2$ and $N_\pi > n_\pi$. So we may apply the results in [11]. Suppose that ψ_E has conductor $\mathfrak{o}_E$. Let v be the newform for π such that $W_v(e) = 1$. Then by Proposition 3.4 and [11, Proposition 5.12], we see that $Z(s, W_v, \Phi_{N_\pi})$ has the form $1/P(q^{-2s})$, where $P(X)$ is a polynomial in $\mathbb{C}[X]$ such that $P(0) = 1$ and $\deg P(X) \leq 1$. So Proposition 6.12 implies that $Z(s, W_v, \Phi_{N_\pi}) = L_E(s, \mu_1)$. Now the assertion follows from Theorem 3.6.   □

8 L-factors of the depth zero supercuspidal representations

In this section, we determine L-factors of the generic depth zero supercuspidal representations of G.

8.1 Regular cuspidal representations of U(2, 1)(k_F)

For any subgroup S of $G = \mathrm{U}(2, 1)(F)$, we denote by $\overline{S}$ the subgroup of $\overline{G} = \mathrm{U}(2, 1)(k_F)$ which corresponds to S. For example, $\overline{B}$ is the Borel subgroup of $\overline{G}$ consisting of the upper triangular elements with unipotent radical $\overline{U}$, and

$$\overline{H} = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in \overline{G} \right\} \simeq \mathrm{U}(1, 1)(k_F).$$

Since k_F is a finite field, we have $\overline{G} = \mathrm{U}(2, 1)(k_F) \simeq \mathrm{U}(3)(k_F)$ and $H \simeq \mathrm{U}(2)(k_F)$. In [6], Ennola classified the irreducible representations of these two groups by giving the character tables.

Let τ be a representation of $\overline{G}$. For any subgroup $\overline{S}$ of $\overline{G}$, we denote by $\tau^{\overline{S}}$ the space of $\overline{S}$-fixed vectors in τ. A representation τ of $\overline{G}$ is called cuspidal if $\tau^{\overline{U}} = \{0\}$. We say that τ is regular when $\mathrm{Hom}_{\overline{U}}(\tau, \overline{\psi}) \neq \{0\}$, where $\overline{\psi}$ is a non-degenerate character of $\overline{U}$. If τ is irreducible and regular, then the space $\mathrm{Hom}_{\overline{U}}(\tau, \overline{\psi})$ is one-dimensional.

For any representation τ of $\overline{G}$, we denote by χ_τ the character of τ. We use the notation of irreducible characters of $\overline{G}$ in [6]. There are the following three kinds of irreducible cuspidal characters.

(C1) $\chi_{q^2-q}^{(t)}$, $1 \leq t \leq q + 1$. The corresponding representation is $(q^2 - q)$-dimensional and non-regular.
(C2) $\chi_{(q-1)(q^2-q+1)}^{(t,u,v)}$, $1 \leq t < u < v \leq q + 1$. The corresponding representation is $(q - 1)(q^2 - q + 1)$-dimensional and regular.
(C3) $\chi_{(q+1)(q^2-1)}^{(t)}$, $1 \leq t \leq q^3, t \not\equiv 0 \pmod{q^2 - q + 1}$. The corresponding representation is $(q + 1)(q^2 - 1)$-dimensional and regular.

Proposition 8.1 *Let τ be an irreducible regular cuspidal representation of $\overline{G}$.*

(i) $\dim \tau^{\overline{B_H}} = 1$.
(ii) $\dim \tau^{\overline{H}} \leq 1$. *The equality holds if and only if $\chi_\tau = \chi_{(q-1)(q^2-q+1)}^{(t,u,q+1)}$ for some $1 \leq t < u < q + 1$.*

Proof (i) Since τ is regular, the restriction of τ to $\overline{U}$ contains a non-degenerate character $\overline{\psi}$. The group $\overline{U_H}$ lies in the kernel of $\overline{\psi}$, so τ has a non-zero $\overline{U_H}$-fixed vector. Because $\overline{U_H}$ is a normal subgroup of $\overline{U}$, the group $\overline{U}$ acts on the space $\tau^{\overline{U_H}}$. We regard $\tau^{\overline{U_H}}$ as $\overline{U}$-module. Then $\tau^{\overline{U_H}}$ is a sum of one-dimensional representations of $\overline{U}$ since $\overline{U}/\overline{U_H}$ is abelian. The cuspidality of τ implies that $\tau^{\overline{U_H}}$ is a sum of non-degenerate characters of $\overline{U}$. Since the diagonal subgroup $\overline{T_H}$ of $\overline{H}$ acts transitively on the set of the non-degenerate characters of $\overline{U}$, every non-degenerate character of $\overline{U}$ occurs in $\tau^{\overline{U_H}}$. Recall that for any non-degenerate character $\overline{\psi}$ of $\overline{U}$, we have $\dim \mathrm{Hom}_{\overline{\psi}}(\tau, \overline{\psi}) = 1$. Thus, every $\overline{\psi}$ occurs in $\tau^{\overline{U_H}}$ with multiplicity one. We fix a non-degenerate character $\overline{\psi}$ of $\overline{U}$ and take a vector v in $\tau^{\overline{U_H}}$ so that $\overline{U}$ acts on $\mathbf{C}v$ by $\overline{\psi}$. Then for any $a \in k_E^\times$, the group $\overline{U}$ acts on $\mathbf{C}\tau(t(a^{-1}))v$ by $\overline{\psi}_a$, where

$$\overline{\psi}_a(u) = \overline{\psi}(t(a)ut(a^{-1})), \quad u \in \overline{U}.$$

We therefore have $\tau^{\overline{U_H}} = \bigoplus_{a \in k_E^\times} \mathbf{C}\tau(t(a^{-1}))v$. Note that $\overline{B_H} = \overline{T_H}\cdot\overline{U_H}$ and $k_E^\times \simeq \overline{T_H}$; $a \to t(a)$. If we take a generator b of a cyclic group $k_E^\times$, then we see that $\sum_{0 \le i \le q^2-2} \tau(b^i)v$ is a basis for $\tau^{\overline{B_H}}$. So we conclude that $\dim \tau^{\overline{B_H}} = 1$.

(ii) Since $\overline{B_H} \subset \overline{H}$, we have $\tau^{\overline{H}} \subset \tau^{\overline{B_H}}$. By (i), we get $\dim \tau^{\overline{H}} \le \dim \tau^{\overline{B_H}} = 1$. For any two class functions χ_1 and χ_2 of $\overline{H}$, we define

$$(\chi_1, \chi_2) = \frac{1}{|\overline{H}|} \sum_{h \in \overline{H}} \chi_1(h)\overline{\chi_2(h)}.$$

We denote by $\mathbf{1}_{\overline{H}}$ the trivial representation of $\overline{H}$. By Schur orthogonality relations, we have $(\chi_\tau, \mathbf{1}_{\overline{H}}) = \frac{1}{|\overline{H}|}\sum_{h \in \overline{H}} \chi_\tau(h) = \dim \tau^{\overline{H}}$. Using the character table in [6], one can check that

$$(\chi_{(q-1)(q^2-q+1)}^{(t,u,q+1)}, \mathbf{1}_{\overline{H}}) = 1, \text{ for } 1 \le t < u < q+1$$

and that $(\chi, \mathbf{1}_{\overline{H}}) = 0$ for any other irreducible regular cuspidal characters of $\overline{G}$. This shows the assertion. $\qquad\square$

Let τ be an irreducible regular cuspidal representation of $\overline{G}$. By Proposition 8.1 (i), we have $\dim \tau^{\overline{B_H}} = 1$. We fix a non-zero element v_0 in $\tau^{\overline{B_H}}$ and consider the vector

$$\sum_{u \in \overline{U_H}} \tau(uw)v_0 = \sum_{a \in k_F} \tau(u(0,a)w)v_0,$$

where

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ u(0,a) = \begin{pmatrix} 1 & 0 & a\sqrt{\epsilon} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \overline{G}$$

and we are regarding $\sqrt{\epsilon} \in \mathfrak{o}_E$ as an element in $k_E = \mathfrak{o}_E/\mathfrak{p}_E$. One may check that $\sum_{a \in k_F} \tau(u(0,a)w)v_0$ also belongs to $\tau^{\overline{B_H}}$. Since $\tau^{\overline{B_H}}$ is one-dimensional, there exists an element α in $\mathbf{C}$ such that

$$\sum_{a \in k_F} \tau(u(0,a)w)v_0 = \alpha v_0.$$

The following lemma determines α.

Lemma 8.2 *We have $\alpha = q$ if $\tau^{\overline{H}} \ne \{0\}$, and $\alpha = -1$ if $\tau^{\overline{H}} = \{0\}$.*

Proof Note that we have $\tau^{\overline{H}} \subset \tau^{\overline{B_H}}$ because $\overline{B_H} \subset \overline{H}$. Suppose that $\tau^{\overline{H}} \ne \{0\}$. Then it follows from Proposition 8.1 (i) that v_0 lies in $\tau^{\overline{H}} = \tau^{\overline{B_H}}$. So we have $\sum_{a \in k_F} \tau(u(0,a)w)v_0 = qv_0$ and $\alpha = q$ since $u(0,a)w \in \overline{H}$.

Suppose that $\tau^{\overline{H}} = \{0\}$. Then we get

$$\alpha\tau(w)v_0 = \sum_{a \in k_F} \tau(wu(0,a)w)v_0 = v_0 + \sum_{a \ne 0} \tau(wu(0,a)w)v_0.$$

For $0 \ne a \in k_F$, we have

$$wu(0,a)w = u(0,(a\epsilon)^{-1})w \begin{pmatrix} a\sqrt{\epsilon} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -(a\sqrt{\epsilon})^{-1} \end{pmatrix} u(0,(a\epsilon)^{-1}).$$

Since v_0 is fixed by $\overline{B_H}$, we obtain

$$\alpha\tau(w)v_0 = v_0 + \sum_{a \neq 0} \tau(u(0, (a\varepsilon)^{-1})w)v_0$$

$$= v_0 + \sum_{a \in k_F} \tau(u(0, a)w)v_0 - \tau(w)v_0$$

$$= v_0 + \alpha v_0 - \tau(w)v_0,$$

and hence $(\alpha + 1)\tau(w)v_0 = (\alpha + 1)v_0$. If $\alpha + 1 \neq 0$, then by an Iwasawa decomposition of $\overline{H}$, we see that v_0 is fixed by $\overline{H}$. This contradicts the assumption that $\tau^{\overline{H}} = \{0\}$. So we conclude that $\alpha = -1$ as required. □

8.2 Generic depth zero supercuspidal representations

Every irreducible depth zero supercuspidal representation of G is induced from a maximal compact subgroup P_0 of G. Up to conjugation, there are two maximal compact subgroup of G. Thus we may assume $P_0 = K_0 = \mathrm{GL}_3(\mathfrak{o}_E) \cap G$ or

$$Z_0 K_1 = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G.$$

We denote by P_1 the pro-p radical of P_0. Then P_0/P_1 is a reductive group over k_F. We note that if $P_0 = \mathrm{GL}_3(\mathfrak{o}_E) \cap G$, then $P_1 = (1 + M_3(\mathfrak{p}_E)) \cap G$ and P_0/P_1 is isomorphic to $\overline{G} = \mathrm{U}(2, 1)(k_F)$. If $P_0 = Z_0 K_1$, then we have

$$P_1 = \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap G$$

and $P_0/P_1 \simeq \mathrm{U}(1, 1)(k_F) \times \mathrm{U}(1)(k_F)$. Let $\overline{\rho}$ be an irreducible cuspidal representation of P_0/P_1. Then $\pi = c\text{-Ind}_{P_0}^G \rho$ is an irreducible depth zero supercuspidal representation of G, where ρ stands for the inflation of $\overline{\rho}$ to P_0. We note that every irreducible depth zero supercuspidal representation of G is obtained in this way.

Proposition 8.3 *With the notation as above, $\pi = c\text{-Ind}_{P_0}^G \rho$ is generic if and only if $P_0 = K_0$ and $\overline{\rho}$ is regular.*

Proof The proof is exactly same as that of Proposition 2.2 in [2]. □

Lemma 8.4 *For $i = 1, 2$, let $\pi_i = c\text{-Ind}_{K_0}^G \rho_i$ be an irreducible depth zero supercuspidal representation of G, where ρ_i is the inflation of an irreducible cuspidal representation $\overline{\rho_i}$ of $\overline{G}$. Suppose that π_1 is isomorphic to π_2. Then $\overline{\rho_1}$ is isomorphic to $\overline{\rho_2}$.*

Proof Suppose that π_1 is isomorphic to π_2. Since ρ_i is contained in the restriction of π_i to K_0, there is an element g in $K_0 \backslash G / K_0$ such that

$$\mathrm{Hom}_{{}^g K_0 \cap K_0}({}^g \rho_1, \rho_2) \neq \{0\}$$

where ${}^g K_0 = g K_0 g^{-1}$ and ${}^g \rho_1$ is the representation of ${}^g K_0$ defined by ${}^g \rho_1(k) = \rho_1(g^{-1}kg)$, $k \in {}^g K_0$. By a Cartan decomposition $G = \bigcup_{i \geq 0} K_0 \zeta^i K_0$, we may assume $g = \zeta^i$. Suppose that $i > 0$. Then we have $\hat{U}(\mathfrak{o}_E) \subset {}^g K_0 \cap K_0$. Observe that $g^{-1}\hat{U}(\mathfrak{o}_E)g \subset (1 + M_3(\mathfrak{p}_E)) \cap G$.

This implies that $\hat{U}(\mathfrak{o}_E)$ lies in the kernel of ${}^g\rho_1$. By $\mathrm{Hom}_{{}^g K_0 \cap K_0}({}^g\rho_1, \rho_2) \neq \{0\}$, the representation ρ_2 has a non-zero $\hat{U}(\mathfrak{o}_E)$-fixed vector. This contradicts the cuspidality of $\overline{\rho_2}$ because the image of $\hat{U}(\mathfrak{o}_E)$ in $\overline{G} = K_0/P_1$ is a maximal unipotent radical. Therefore we conclude that $i = 0$ and $g = \zeta^0 = 1$. Since $\mathrm{Hom}_{K_0}(\rho_1, \rho_2) \neq \{0\}$ and ρ_1, ρ_2 are irreducible, we have $\rho_1 \simeq \rho_2$, and hence $\overline{\rho_1} \simeq \overline{\rho_2}$, as required. $\qquad\square$

8.3 Conductors of depth zero supercuspidal representations

From now on, we assume that $P_0 = K_0 = \mathrm{GL}_3(\mathfrak{o}_E) \cap G$, $P_1 = (1 + M_3(\mathfrak{p}_E)) \cap G$, and $\overline{\rho}$ is an irreducible regular cuspidal representation of $\overline{G} \simeq K_0/P_1$. Then $\pi = c\text{-Ind}_{K_0}^G \rho$ is an irreducible generic depth zero supercuspidal representation of G.

Proposition 8.5 *Let* $\pi = c\text{-Ind}_{K_0}^G \rho$ *be an irreducible generic depth zero supercuspidal representation of G as above.*

 (i) $2 \leq N_\pi \leq 3$.
(ii) $N_\pi = 2$ *if and only if* $\overline{\rho}$ *has a non-zero* $\overline{H}$*-fixed vector.*

Proof (i) By [12, Corollary 5.5 (i)], we have $2 \leq N_\pi$. Observe

$$K_0 \cap \zeta K_3 \zeta^{-1} = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^2 & 1 + \mathfrak{p}_E^3 & \mathfrak{p}_E \\ \mathfrak{p}_E & \mathfrak{p}_E^2 & \mathfrak{o}_E \end{pmatrix} \cap G.$$

This implies that the image of $K_0 \cap \zeta K_3 \zeta^{-1}$ in K_0/P_1 is $\overline{B_H}$. By Proposition 8.1 (i), we can take a non-zero $\overline{B_H}$-fixed vector v_0 in $\overline{\rho}$. We regard v_0 as a non-zero $K_0 \cap \zeta K_3 \zeta^{-1}$-fixed vector in ρ. Then the function

$$f(g) = \begin{cases} \rho(p)v_0, & \text{for } g = p\zeta k, \ p \in K_0, \ k \in K_3, \\ 0, & otherwise \end{cases}$$

is well-defined. Since f is a non-zero K_3-fixed vector in $\pi = c\text{-Ind}_{K_0}^G \rho$, we obtain $N_\pi \leq 3$.

   (ii) Since we have seen that $2 \leq N_\pi$, it suffices to show that $V(2) \neq \{0\}$ if and only if $\overline{\rho}$ has a non-zero $\overline{H}$-fixed vector. Suppose that $\overline{\rho}$ has a non-zero $\overline{H}$-fixed vector v_0. Then we can construct a non-zero function f in $V(2)$ as follows: One may check that

$$K_0 \cap \zeta K_2 \zeta^{-1} = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E^2 & \mathfrak{p}_E \\ \mathfrak{o}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G.$$

Therefore the image of $K_0 \cap \zeta K_2 \zeta^{-1}$ in K_0/P_1 is $\overline{H}$. Regarding v_0 as a $K_0 \cap \zeta K_2 \zeta^{-1}$-fixed vector in ρ, we can define a non-zero function f in $V(2)$ by

$$f(g) = \begin{cases} \rho(p)v_0, & \text{for } g = p\zeta k, \ p \in K_0, \ k \in K_2, \\ 0, & otherwise. \end{cases}$$

This implies $V(2) \neq \{0\}$.

   Suppose that $V(2) \neq \{0\}$. Then π has a non-zero K_2-fixed vector v. Since we are assuming that π is of depth zero, there exists a non-zero $(1 + M_3(\mathfrak{p}_E)) \cap G$-fixed vector in π. This implies that the group $Z_1 = (1 + \mathfrak{p}_E) \cap Z$ acts trivially on π. So we see that $\pi(\zeta)v$ is fixed by

$$Z_1 \cdot \zeta K_2 \zeta^{-1} = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{o}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G.$$

Since $P_1 \subset Z_1 \cdot \zeta K_2 \zeta^{-1}$, the vector $\pi(\zeta)v$ lies in π^{P_1}. If we regard π^{P_1} as a K_0/P_1-module, π^{P_1} is decomposed into a sum of irreducible cuspidal representations of $\overline{G} \simeq K_0/P_1$ because of Theorem 8.9 in [13]. By Lemma 8.4 and a standard argument, one can show that π^{P_1} is isomorphic to $\overline{\rho}$. Since the image of $Z_1 \cdot \zeta K_2 \zeta^{-1}$ in K_0/P_1 is $\overline{H}$, $\pi(\zeta)v$ is a non-zero $\overline{H}$-fixed vector $\pi^{P_1} \simeq \overline{\rho}$. This completes the proof.   □

Remark 8.6 Let $\pi = c\text{-Ind}_{K_0}^G \rho$ be an irreducible generic depth zero supercuspidal representation of G. Recall that $\overline{\rho}^{\overline{H}} \subset \overline{\rho}^{\overline{B_H}}$. By the proof of Proposition 8.5, a newform f for π is given by

$$f(g) = \begin{cases} \rho(p)v_0, & g = p\zeta k, \ p \in K_0, \ k \in K_{N_\pi}, \\ 0, & otherwise, \end{cases}$$

where v_0 is a non-zero $\overline{B_H}$-fixed vector in $\overline{\rho}$.

8.4 *L*-factors of depth zero supercuspidal representations

We recall from [11] a formula of *L*-factors $L(s, \pi)$ of irreducible generic supercuspidal representations (π, V). We abbreviate $N = N_\pi$. Let λ be the eigenvalue of the Hecke operator T on $V(N)$. By Theorem 4.2, Propositions 3.5 and 5.12 in [11], we have

$$L(s, \pi) = \frac{1}{1 - (\lambda + q^2)q^{-2-2s}}. \tag{8.7}$$

We further recall from [11] a description of λ. Let $\delta : V(N) \to V(N-1)$ be the level lowering operator. For $v \in V(N)$, it follows from Lemmas 5.4 and 5.9 in [11] that

$$\lambda v = T v = \pi(\zeta^{-1})\delta v - \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \pi(u(y, 0))v$$
$$+ \sum_{a \in \mathfrak{o}_E/\mathfrak{p}_E} \sum_{b \in \mathfrak{p}_F^{-N}/\mathfrak{p}_F^{2-N}} \pi(u(a, b)\zeta)v.$$

Since $V(N-1) = \{0\}$, we have $\delta v = 0$ so that

$$\lambda v = - \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \pi(u(y, 0))v + \sum_{a \in \mathfrak{o}_E/\mathfrak{p}_E} \sum_{b \in \mathfrak{p}_F^{-N}/\mathfrak{p}_F^{2-N}} \pi(u(a, b)\zeta)v.$$

We assume that $\pi = c\text{-Ind}_{K_0}^G \rho$ is an irreducible generic depth zero supercuspidal representation of G. Take $f \in V(N)$ as in Remark 8.6. Then we obtain

$$\lambda f(\zeta) = - \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} f(\zeta u(y, 0)) + \sum_{a \in \mathfrak{o}_E/\mathfrak{p}_E} \sum_{b \in \mathfrak{p}_F^{-N}/\mathfrak{p}_F^{2-N}} f(\zeta u(a, b)\zeta).$$

Note that $f(\zeta)$ is a non-zero element in $\overline{\rho}^{\overline{B_H}}$. Since $\zeta u(y, 0) = \zeta u(y, 0)\zeta^{-1}\zeta = u(\varpi y, 0)\zeta$, we have

$$\sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} f(\zeta u(y, 0)) = \sum_{y \in \mathfrak{o}_E/\mathfrak{p}_E} f(u(y, 0)\zeta) = \sum_{y \in \mathfrak{o}_E/\mathfrak{p}_E} \rho(u(y, 0))f(\zeta).$$

Since we are assuming that $\overline{\rho}$ is cuspidal, we have $\overline{\rho}^{\overline{U}} = \{0\}$. Because $f(\zeta) \in \overline{\rho}^{\overline{B_H}} \subset \overline{\rho}^{\overline{U_H}}$ and $\overline{U}$ normalizes $\overline{U_H}$, we see that $\sum_{y \in \mathfrak{o}_E/\mathfrak{p}_E} \rho(u(y, 0))f(\zeta)$ lies in $\overline{\rho}^{\overline{U}} = \{0\}$. So we obtain $\sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} f(\zeta u(y, 0)) = 0$, and hence

$$\lambda f(\zeta) = \sum_{a \in \mathfrak{o}_E / \mathfrak{p}_E} \sum_{b \in \mathfrak{p}_F^{-N} / \mathfrak{p}_F^{2-N}} f(\zeta u(a,b)\zeta).$$

By $\zeta u(a,b)\zeta = \zeta u(a,0)\zeta^{-1}\zeta u(0,b)\zeta = u(\varpi a, 0)\zeta u(0,b)\zeta$ and $u(\varpi a, 0) \in (1 + M_3(\mathfrak{p}_E)) \cap G \subset \ker \rho$, we have

$$f(\zeta u(a,b)\zeta) = \rho(u(\varpi a, 0)) f(\zeta u(0,b)\zeta) = f(\zeta u(0,b)\zeta)$$

and

$$\lambda f(\zeta) = q^2 \sum_{b \in \mathfrak{p}_F^{-N} / \mathfrak{p}_F^{2-N}} f(\zeta u(0,b)\zeta). \tag{8.8}$$

Proposition 8.9 *Let* $\pi = c\text{-Ind}_{K_0}^G \rho$ *be an irreducible generic depth zero supercuspidal representation of* G.

(i) *If* $N_\pi = 2$, *then we have* $L(s,\pi) = L_E(s, \mathbf{1})$.
(ii) *If* $N_\pi = 3$, *then* $L(s,\pi) = 1$.

Proof (i) Suppose that $N = N_\pi = 2$. Then, by Remark 8.6, the support of $f \in V(2)$ is $K_0 \zeta K_2$. Observe that

$$K_0 \zeta K_2 \subset \begin{pmatrix} \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \end{pmatrix}.$$

For any $b \in \mathfrak{p}_F^{-N} / \mathfrak{p}_F^{2-N}$, the $(3,3)$-entry of $\zeta u(0,b)\zeta$ is ϖ^{-2}. So we have $\zeta u(0,b)\zeta \notin K_0 \zeta K_2$ and $f(\zeta u(0,b)\zeta) = 0$. By (8.8), we get $\lambda f(\zeta) = 0$ and $\lambda = 0$. Therefore it follows from (8.7) that $L(s,\pi) = \frac{1}{1-q^{-2s}} = L_E(s, \mathbf{1})$.

(ii) If $N = N_\pi = 3$, then Remark 8.6 implies that the support of f is $K_0 \zeta K_3$. One may check that

$$K_0 \zeta K_3 \subset \begin{pmatrix} \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-2} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-2} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-2} \end{pmatrix}.$$

We see that if $b \notin \mathfrak{p}_F^{-2}$, then $\zeta u(0,b)\zeta \notin K_0 \zeta K_3$ and hence $f(\zeta u(0,b)\zeta) = 0$. Therefore we have

$$\lambda f(\zeta) = q^2 \sum_{b \in \mathfrak{p}_F^{-2} / \mathfrak{p}_F^{-1}} f(\zeta u(0,b)\zeta)$$

by (8.8). Since $\zeta u(0,b)\zeta = \zeta u(0,b)\zeta^{-1}\zeta^2 = u(0, \varpi^2 b)\zeta^2$, we get

$$\lambda f(\zeta) = q^2 \sum_{b \in \mathfrak{o}_F / \mathfrak{p}_F} \rho(u(0,b)) f(\zeta^2).$$

Note that f is fixed by K_3 and ζ^2 lies in $w\zeta K_3$. Thus we have

$$\lambda f(\zeta) = q^2 \sum_{b \in \mathfrak{o}_F / \mathfrak{p}_F} \rho(u(0,b)) f(w\zeta) = q^2 \sum_{b \in \mathfrak{o}_F / \mathfrak{p}_F} \rho(u(0,b)w) f(\zeta).$$

Recall that $f(\zeta)$ is a non-zero element in $\overline{\rho}^{B_H}$. By the assumption that $N_\pi = 3$ and Proposition 8.5 (ii), we have $\overline{\rho}^H = \{0\}$. So Lemma 8.2 implies $\lambda f(\zeta) = -q^2 f(\zeta)$ and $\lambda = -q^2$. We conclude $L(s,\pi) = 1$ because of (8.7). $\qquad\square$

Remark 8.10 Let π be an irreducible generic supercuspidal representation of G. It follows from Proposition 4.2 (i) that $L(s, \pi) = 1$ or $L_E(s, \mathbf{1})$. Propositions 8.1, 8.5 and 8.9 imply that both cases occur.

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