LECTURE 9 — EULERIAN INTEGRAL REPRESENTATIONS

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Abstract. We introduce the Rankin-Selberg integral representations of L-functions of automorphic forms, generalizing the integral representation of the Riemann zeta function. This representations allows to uncover the key properties of L-functions: they extend meromorphically on the whole complex plane, are bounded in vertical strips and satisfy a function equation relating s and $1-s$.

Let k be a number field. Let (π, V_π) (resp $(\pi', V_{\pi'})$) be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ (resp. $\mathrm{GL}_m(\mathbf{A})$). The aim of this talk is to describe how to define the Eulerian integrals $I(s,\pi,\pi')$ for $s \in \mathbf{C}$, and how they relate to the attached L-functions $L(s, \pi \times \pi')$.

1. Case $\mathrm{GL}_2 \times \mathrm{GL}_1$

This is the case $n = 2$ and $m = 1$. Let (π, V_π) be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$. Let $\chi : \mathbf{A}^\times/k^\times \to \mathbf{C}^\times$ a Hecke character.

To a cuspidal modular form f and for $\Re(s)$ large enough, we attach the associated L-function that may be defined as

$$L(s,f) = \int_0^\infty f(iy) y^s \frac{\mathrm{d}y}{y}. \tag{1}$$

Replacing f by its Fourier expansion $\sum a_n e(nz)$, we recover the classical definition as the Dirichlet series

$$L(s,f) = \sum_{n \geqslant 1} \frac{a_n}{n^s}. \tag{2}$$

It is possible to attach such an L-function to a cuspidal automorphic representation (π_f, V_f) of $\mathrm{GL}_2(\mathbf{A})$ (more precisely to its automorphic adelization ϕ_f). The above indeed rewrites

$$L(s,f) = \int_0^\infty \underbrace{f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} i\right)}_{=\phi_f \begin{pmatrix} y & \\ & 1 \end{pmatrix}} y^s \frac{\mathrm{d}y}{y}. \tag{3}$$

Define more generally, for any cuspidal $\phi \in V_\pi$,

$$I(s,\phi,\chi) = \int_{k^\times \backslash \mathbf{A}^\times} \phi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi(a) |a|^{s-\frac{1}{2}} \mathrm{d}^\times a. \tag{4}$$

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These notes have been taken on the go by Didier Lesesvre. They may contain typos and errors.

Proposition. *The integral $I(s,\phi,\chi)$ satisfies the following properties:*

- *$I(s,\phi,\chi)$ is absolutely convergent for $\Re(s)$ large enough, and admits a holomorphic continuation to the whole complex plane*
- *$I(s,\phi,\chi)$ it is bounded in vertical strips*
- *We have the functional equation*
$$I(s,\phi,\chi) = I(1-s,\tilde{\phi},\chi^{-1}) \tag{5}$$
where $\tilde{\phi}(g) = \phi({}^t g^{-1}) =: \phi(g^\iota)$.

Consider the tensor product decomposition $\pi \simeq \otimes'_v \pi_v$, and under this isomorphism $\phi \mapsto \otimes_v \phi_v$. But then, even though ϕ is not really decomposable, the Whittaker model is (cf. previous lecture) and we can write $W_\phi(g) = \prod_v W_{\phi_v}(g_v)$ for all $g \in \mathrm{GL}_2(\mathbf{A})$.

Replacing ϕ by its Fourier transform we get
$$\Phi(g) = \sum_{\gamma\in k} W_\phi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) \tag{6}$$
so the zeta integral above rewrites
$$I(s,\phi,\chi) = \sum_{\gamma\in k^\times}\int_{k^\times\backslash\mathbf{A}^\times} W_\phi\begin{pmatrix}\gamma a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\mathrm{d}^\times a \tag{7}$$
but we notice that $|a| = |\gamma a|$ and $\chi(a) = \chi(a\gamma)$ since $\gamma \in k^\times$, so we obtain
$$I(s,\phi,\chi) = \int_{\mathbf{A}^\times} W_\phi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\mathrm{d}^\times a. \tag{8}$$
By the above factorization, this splits into
$$I(s,\phi,\chi) = \prod_v \int_{k_v} W_{\xi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\chi_v(a_v)|a_v|^{s-1/2}\mathrm{d}^\times a_v =: \prod_v Z_\pi(s, W_{\xi_v},\chi_v) \tag{9}$$
for $\Re(s)$ large enough. Hence, the global zeta integral $I(s,\phi,\chi)$ splits into a Euler product of "local zeta integrals".

2. $\mathrm{GL}_n \times \mathrm{GL}_n$ for $m < n$

Let $m < n$. Let ϕ be a cusp form in a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_n(\mathbf{A})$ and ϕ' be a cusp form in a cuspidal automorphic representation $(\pi', V_{\pi'})$ of $\mathrm{GL}_m(\mathbf{A})$. Consider the maximal unipotent subgroup
$$N_k = \begin{pmatrix} 1 & \star \\ & 1\end{pmatrix} \subset \mathrm{GL}_k \tag{10}$$
and the mirabolic subgroup
$$P_k = \begin{pmatrix} \star & \star \\ 0 & 1\end{pmatrix} \tag{11}$$
where the top left $\star$ is in M_{n-1}. Introduce then
$$Y = \begin{pmatrix} I_{m+1} & \star \\ 0 & x\end{pmatrix} \subset N_n \tag{12}$$
for $x \in N_{n-m-1}$. Define then, for $\phi \in V_\pi$ and ψ an additive character of $k\backslash\mathbf{A}$, a projection $\mathbb{P}\phi : P_{m+1}(\mathbf{A}) \to \mathbf{C}$ by
$$\mathbb{P}\phi(p) : |\det p|^{-\frac{n-m-1}{2}} \int_{[Y]} \phi\left(y\begin{pmatrix} P & 0 \\ 0 & I_{n-m-1}\end{pmatrix}\right)\psi^{-1}(y)\mathrm{d}y. \tag{13}$$

Note that, for $m = n-1$, the projection is merely the restriction map to P_n. We recover what happened in the previous cas $\mathrm{GL}_2 \times \mathrm{GL}_1$ (or more generally on $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$): the integrand is simply the product of both automorphic forms.

Question. Why is the $n = m$ case of Rankin-Selberg different? Where is the crux of $m < n$, where do we see its effect? The crux is that if we define the Eulerian integral by

$$\int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \phi(g)\phi'(g)|\mathrm{det} g|^s dg$$

for $\phi \in V_\pi \subset \mathcal{A}_0(\mathrm{GL}_n)$ and $\phi' \in V_{\pi'} \subset \mathcal{A}_0(\mathrm{GL}_n)$, then, besides the convergence issues, would give an invariant pairing and would be zero unless $\tilde{\pi} \cong \pi' \otimes |\mathrm{det}|^s$.

Proposition. *The projection $\mathbb{P}\phi$ is left-invariant under $P_{m-1}(k)$ and cuspidal on $P_{m+1}(\mathbf{A})$ (for all relevant unipotents). Moreover, we have the Fourier expansion*

$$\mathbb{P}\phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = |\det h|^{-\frac{n-m-1}{2}} \sum_{\gamma \in N_m(k)\backslash \mathrm{GL}_m(k)} W_\phi \begin{pmatrix} \gamma h & \\ & I_{n-m} \end{pmatrix}. \tag{14}$$

For $\phi \in V_\pi$ (on GL_n), we then have a map $\mathbb{P}\phi : P_{m-1}(\mathbf{A}) \to \mathbf{C}$. We set

$$I(s, \phi, \phi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbf{A})} \mathbb{P}\phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi'(h)|\det h|^{s-\frac{1}{2}} \mathrm{d}h. \tag{15}$$

Proposition. *The integral $I(s, \phi, \phi')$ satisfies the following properties:*

- *$I(s, \phi, \phi')$ is convergent for $Re(s) \gg 1$, and admits a holomorphic continuation to the whole complex plane*
- *$I(s, \phi, \phi')$ is bounded in vertical strips*
- *We have the functional equation*
$$I(s, \phi, \phi') = \tilde{I}(1-s, \tilde{\phi}, \tilde{\phi}') \tag{16}$$
where $\tilde{\phi}(g) = \phi(g^\iota)$ and $\tilde{I}$ is a "tilded version of I".

Now consider factorizations of π and π' (more precisely, up to isomorphism, or truly for their Whittaker models). Then for $\Re(s)$ large enough we have

$$I(s, \phi, \phi') = \prod_v Z_v(s, W_{\phi_v}, W_{\phi'_v}) \tag{17}$$

where the local integrals are given by (the global integral is the same thing with $\mathbf{A}$'s)

$$Z_v(s, W_{\phi_v}, W_{\phi'_v}) = \int_{N_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_v \begin{pmatrix} h_v & \\ & I_{n-m} \end{pmatrix} W'_v(h_v)|\det h_v|_v^{s-\frac{n-m}{2}} \mathrm{d}h_v. \tag{18}$$

Question. Write the computation of top page 40 of Cogdell. The W'_ϕ does not come from the Fourier expansion, but from a change of variables and by definition. This is very interesting.

3. $\mathrm{GL}_n \times \mathrm{GL}_n$

Here we do need to weight by an extra Schwartz-Bruhat function $\Phi \in \mathcal{S}(\mathbf{A}^n)$. Assume it decomposes as a restricted tensor product $\otimes'_v \Phi_v$. On Archimedean places, Φ_v is smooth (i.e. C^∞) and rapidly decreasing, while at finite places Φ_v is smooth (locally constant) and compactly supported.

Introduce the theta series, for η a Hecke character,

$$\Theta_\phi(a,g)=\sum_{\xi\in k^n}\Phi(a\xi g) \tag{19}$$

for $a\in k^\times$ and $g\in \mathrm{GL}_n(\mathbf{A})$. Define the Eisenstein series

$$E(g,s)=E(g,s,\phi,\eta)=|\det g|^s\int_{k^\times\backslash\mathbf{A}^\times}\Theta'_\Phi(a,g)\eta(a)|a|^{ns}\mathrm{d}^\times a \tag{20}$$

where $\Theta'_\Phi=\Theta_\Phi-\Phi(0)$. We removed the singularity of Θ_Φ here, so that each term in the sum is rapidly decreasing, and this fact ensures convergence of the integral below (exactly as it happens for the classical Riemann zeta function). The basic analytic properties of this Eisenstein series are the following.

Proposition. i) *$E(g,s)\in\mathcal{A}^\infty(\eta^{-1})$, i.e., $E(g,s)$ is a smooth automorphic form on $\mathrm{GL}_n(\mathbb{A})$ of central character η^{-1}.*

ii) *$E(g,s)$ extends to a meromorphic function of s with at most simple poles at $s=i\sigma$ and $s=1+i\sigma$ with $\sigma\in\mathbf{R}$ such that $\eta(a)=|a|^{-in\sigma}$,*

iii) *$E(g,s)$ is bounded in vertical strips away from its poles.*

iv) *$E(g,s)$ satisfies a functional equation*

$$E(g,s,\Phi,\eta)=E(g^\iota,1-s,\hat{\Phi},\eta^{-1}),$$

where $g^\iota={}^tg^{-1}$ and $\hat{\Phi}$ is the Fourrier transform of Φ.

Introduce then

$$I(s,\phi,\phi',\Phi)=\int_{Z_n(\mathbf{A})\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbf{A})}\phi(g)\phi'(g)E(g,s,\Phi,\omega_\pi\omega_{\pi'})\mathrm{d}g. \tag{21}$$

Proposition. *The integral $I(s,\phi,\phi',\Phi)$ satisfies the following properties:*

- *$I(s,\phi,\phi',\Phi)$ is convergent for $Re(s)\gg 1$, and admits a meromorphic continuation to the whole complex plane with at most simple poles (totally understood).*
- *$I(s,\phi,\phi',\Phi)$ is bounded in vertical strips*
- *We have the functional equation*

$$I(s,\phi,\phi',\Phi)=\tilde{I}(1-s,\tilde{\phi},\tilde{\phi}',\hat{\Phi}) \tag{22}$$

where $\tilde{\phi}(g)=\phi(g^\iota)$, $\tilde{I}$ is a "tilded version of I" and $\hat{\Phi}$ is a transform of Φ.

We then have

$$I(s,\phi,\phi',\Phi)=\prod_v Z_v(s,W_{\phi_v},W_{\phi'_v},\Phi_v) \tag{23}$$

where, letting $e_n=(0,\dots,0,1)\in k^n$,

$$Z_v(s,W_v,W'_v,\Phi_v)=\int_{N(k_v)\backslash\mathrm{GL}_n(k_v)}W_v(g_v)W'_v(g_v)\Phi(e_ng_v)|\det g_v|_v^s\mathrm{d}g_v. \tag{24}$$

So we also have a splitting in a Euler product in this case.

4. L-functions

In all the cases above, we defined a famile of global integrals $(I(s,\phi,\phi',\Phi))_{\phi,\phi',\Phi}$ and correspond families of local integrals $(Z_v(s,W_v,W'_v,\Phi_v))_{W_v,W'_v,\Phi_v}$.

The crux is that, for almost all places (the finite ones), the family of such $Z_v(s,W_v,W'_v,\Phi_v)$ are all rational functions of q_v^{-s}, and they admit a common denominator. They form in particular a fractional ideal of $\mathbf{C}(q_v^{-s})$, and by principaly they admit a generator. A generator can be chosen by normalizing it so that it is of the form $P(q_v^{-s})^{-1}$ with $P \in \mathbf{C}[X]$ with $P(0) = 1$. We denote

$$L(s, \pi_v \times \pi'_v) := P(q_v^{-s})^{-1} \tag{25}$$

this generator, and call it the L-function attached to $\pi_v \times \pi'_v$. We set in particular

$$L(s, \pi_v) = L(s, \pi_v \times \chi_0) \tag{26}$$

where χ_0 is the trivial character on $k_v^\times$. These notions of L-functions do not (unlike a priori the zeta integrals defined all along this lecture) depend on any specific choices of cusp forms ϕ_v, ϕ'_v or on test-functions Φ_v.

These local L-functions can be put back together to define a global L-function, and this one can then be related to the global zeta integrals defined above. The good analytic properties of the (local or global) L-functions can then be straightforwardly deduced from the properties of the zeta integrals.