LECTURE 8 — WHITTAKER MODELS

TALK BY DIDIER LESESVRE

.

Abstract. In this talk we will start from the Fourier expansion of a classical automorphic cusp form on GL(2), and deduce by induction the Fourier expansion on GL(n). The Fourier coefficients involved in this expansion have particularly interesting properties, and constitute a privileged *model* of the representation. This model has an analogous meaning in the local setting, where they can be proven to be unique. From this very deep result, we deduce many nontrivial consequences for global models and automorphic forms.

Automorphic representations are defined as being simultaneously representations of the group $\mathrm{GL}_n(\mathbf{A}_f)$ and $(\mathfrak{g}_\infty, K_\infty)$-modules, satisfying certain compatibility properties. These are very difficult to manipulate and compute with. Do we have good models (i.e. explicit spaces and actions) to use instead?

1. Fourier expansion for GL(2)

Let (π, V_π) a smooth cuspidal representation, i.e. the space $V_\pi \subset \mathcal{A}_0^\infty$ is a subspace of the space of cuspidal automorphic representations (introduced in Lecture 1). Let $\phi \in V_\pi$ a corresponding cusp form.

We concentrate first on the GL(2)-case, since it contains the stem of the GL(n)-case, yet being more handable. Let N the subgroup of unipotent upper triangular matrices.

1.1. A first expansion.

For $g \in G(\mathbf{A})$, the function

$$\phi_g : x \in \mathbf{A} \longmapsto \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \tag{1}$$

is smooth and k-periodic, by automorphy of ϕ. In other words, it induces a function on $k\backslash\mathbf{A}$. Since k is discrete and $k\backslash\mathbf{A}$ is compact and abelian, we have a Fourier expansion parametrized by the dual of $k\backslash\mathbf{A}$:

$$\phi_g(x) = \sum_{\psi \in \widehat{k\backslash\mathbf{A}}} W_{\phi,\psi}(g)\psi(x) \tag{2}$$

where the Fourier coefficients $W_{\phi,\psi}(g)$ are given by

$$W_{\phi,\psi}(g) = \langle \phi_g, \psi \rangle := \int_{k\backslash\mathbf{A}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi^{-1}(x)\mathrm{d}x. \tag{3}$$

In particular, taking $x = 0$, we deduce the Fourier expansion of ϕ:

$$\phi(g) = \sum_{\psi \in \widehat{k\backslash\mathbf{A}}} W_{\phi,\psi}(g). \tag{4}$$

Date: February 3 and 10, 2022.
Notes updated on February 10, 2022
These notes have been taken on the go by Didier Lesesvre. They may contain typos and errors.

1.2. **Parametrizing by a single orbit of coefficients.** By duality theory, $\widehat{k\backslash \mathbf{A}} \simeq k$: more precisely, fix a nontrivial *additive* character ψ of $k\backslash\mathbf{A}$, and then all the characters of $k\backslash\mathbf{A}$ are parametrized by $\psi_\gamma : x \mapsto \psi(\gamma x)$ for $\gamma \in k$. So the Fourier expansion rewrites

$$\phi(g) = \sum_{\gamma\in k} W_{\phi,\psi_\gamma}(g). \tag{5}$$

But ϕ is cuspidal, so that by definition

$$\int_{k\backslash\mathbf{A}} \phi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) \mathrm{d}x = 0 = W_{\phi,\psi_0}(g). \tag{6}$$

For $\gamma \neq 0$, we change variables and get

$$\begin{aligned} W_{\phi,\psi_\gamma}(g) &= \int_{k\backslash\mathbf{A}} \phi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right)\psi^{-1}(\gamma x)\mathrm{d}x = \int_{k\backslash\mathbf{A}} \phi\left(\begin{pmatrix}1 & \gamma^{-1}x\\ & 1\end{pmatrix} g\right)\psi^{-1}(x)\mathrm{d}x \\ &= \int_{k\backslash\mathbf{A}} \phi\left(\begin{pmatrix}\gamma^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)\psi^{-1}(\gamma x)\mathrm{d}x = W_{\phi,\psi}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right), \end{aligned}$$

and the left factor inside ϕ can be omitted by automorphy of ϕ. We hence obtain a nice form for the Fourier expansion:

$$\phi(g) = \sum_{\gamma\in k^\times} W_{\phi,\psi}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right). \tag{7}$$

A fundamental fact coming from these calculations is that all the Fourier coefficients stem from a single one of them. From now on we can omit the mention to ψ, since any choice of nontrivial ψ would do.

1.3. **Comparison with classical Fourier expansions.** We can make a parallel with the classical Fourier analysis for 1-periodic functions f, where

$$f(x+y) = \sum_{n\in\mathbf{Z}} \widehat{f_x}(n)e(ny) \tag{8}$$

thus

$$f(x) = \sum_{n\in\mathbf{Z}} \widehat{f_x}(n) \tag{9}$$

where

$$\widehat{f_x}(n) = \int_0^1 f(x+y)e(-ny)\mathrm{d}y = \widehat{f}(n)e(nx) =: W_{f,n}(x) \tag{10}$$

These coefficients are endowed with similar properties as those we noticed on the W_phi. We can rewrite the Fourier expansion similarly as

$$f(x) = \sum_{n\in\mathbf{Z}} W_{f,n}(x). \tag{11}$$

Note that we have the property

$$W_{f,n}(k+x) = e(kn)W_{f,n}(x) \tag{12}$$

that is reminiscent of the property of the "automorphic" Fourier coefficients

$$W_{\phi,\psi}(ng) = \psi(n)W_{\phi,\psi}(g). \tag{13}$$

2. Fourier expansion for GL(n)

For the case of GL(n), it works similarly "by induction". It is a bit trickier though, since N is then no more commutative and there are some computational tricks to be used. Let's write it for GL(3) for convenience, and see how it builds from the Fourier expansion on GL(2). The proof will work exactly similarly replacing 3 by n.

Let ϕ be an automorphic cusp form for GL(3). The cuspidality is understood here in the sense that

$$\int_{[N]} \phi(ng)\mathrm{d}g = 0 \tag{14}$$

for *any* upper unipotent subgroup N, not only a maximal one (i.e. they can be attached to any upper parabolic, not only the maximal one). For GL(3), these are (up to conjugation)

$$\begin{pmatrix} 1 & \star & \star \\ & 1 & \star \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_2 & \star \\ & 1 \end{pmatrix}. \tag{15}$$

Let ψ be a continuous character of $k\backslash\mathbf{A}$, if need be inflated to a character of the upper triangular matrices (by $\phi(n) := \psi(n_{1,2} + \cdots + n_{n-1,n})$).

2.1. A first expansion.

In order to be able to apply the results of abelian harmonic analysis, let's consider the function

$$(a, b) \longmapsto \phi\left(\begin{pmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{pmatrix} g\right), \tag{16}$$

the crux here being that the unipotent subgroup chosen for the action is commutative, so that we can apply usual Fourier expansion (typically expanding with respect to the variable a, then do the same with b) and get

$$\phi\left(\begin{pmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{pmatrix} g\right) = \sum_{\beta\in k^2} \underbrace{\int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} 1 & & x \\ & 1 & y \\ & & 1 \end{pmatrix} g\right) \psi(-\beta\cdot(x,y))\mathrm{d}x\mathrm{d}y}_{\phi_\beta(g)} \ \psi(\beta\cdot(a,b))$$

Taking $(a, b) = (0, 0)$, we get a first expansion (with respect to the last column)

$$\phi(g) = \sum_{\beta\in k^2} \phi_\beta(g). \tag{17}$$

2.2. Parametrizing by a single orbit of coefficients.

Note that $\phi_{(0,0)}(g) = 0$ for all g, by cuspidality. All the others $\beta \in k^2\backslash\{(0,0)\}$ are in a single orbit modulo $\mathrm{GL}_2(k)$, say $\beta = (0, 1)\cdot\gamma$ for $\gamma \in \mathrm{GL}_2(k)$ uniquely determined modulo $\mathrm{Stab}(0, 1) = P_2(k) = \left(\begin{smallmatrix} \star & \star \\ 0 & 1 \end{smallmatrix}\right)$. Hence, we get

$$\phi(g) = \sum_{\gamma\in P_2(k)\backslash \mathrm{GL}_2(k)} \phi_{(0,1)\cdot\gamma}(g). \tag{18}$$

Remark. This is a crucial fact in all this process, which was not that obvious to detect in the GL(2)-case. This allows to write all the coefficients as stemming from a single one, making it ultimately a genuine model (it is possible to recover ψ from a single coefficient W_ϕ). This fact critically breaks in the case of other groups.

2.3. **Parametrizing by a single coefficient.** To undertsand better these coefficients, rewrite the integral as follows, changing variables to see the effect of γ in the variable rather than in the character:

$$\phi_{(0,1)\cdot\gamma}(g) = \int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} I & y \\ & 1 \end{pmatrix} g\right) \psi(-(0,1)\cdot\gamma y)\mathrm{d}y \tag{19}$$

$$= \int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} I & \gamma^{-1}y \\ & 1 \end{pmatrix} g\right) \psi(-(0,1)y)\mathrm{d}y \tag{20}$$

$$= \int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} \gamma^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} I & y \\ & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi(-(0,1)y)\mathrm{d}y \tag{21}$$

$$= \int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} I & y \\ & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi(-(0,1)y)\mathrm{d}y \tag{22}$$

$$= \phi_{(0,1)}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right). \tag{23}$$

We hence paramtetrized all the Fourier coefficients from a single one, and the Fourier expansion now rewrites

$$\phi(g) = \sum_{\gamma\in P_2(k)\backslash \mathrm{GL}_2(k)} \phi_{(0,1)}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right). \tag{24}$$

2.4. **Using the Fourier expansion for** $\mathrm{GL}(2)$**.** The function

$$\phi_{g,(0,1)} : \gamma \mapsto \phi_{(0,1)}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \tag{25}$$

is a function on $\mathrm{GL}_2(\mathbf{A})$. Moreover, it is automorphic (by automorphy of ϕ) and cuspidal (by cuspidality of ϕ). Hence, by the above section (for general n, we would use the induction hypothesis), we can write its Fourier expansion

$$\phi_{g,(0,1)}(\gamma) = \sum_{\delta\in N_1(k)\backslash \mathrm{GL}_1(k)} \int_{[N_2]} \phi_{g,(0,1)}\left(u\begin{pmatrix} \delta & \\ & 1 \end{pmatrix}\gamma\right)\psi^{-1}(u)\mathrm{d}u. \tag{26}$$

2.5. **A transfer formula.** The trick is now to rewrite the above integral as an integral over $\mathrm{GL}(3)$ matrices, to obtain the desired Fourier expansion. Let $h = \left(\begin{smallmatrix} \delta & \\ & 1 \end{smallmatrix}\right)\gamma$ for convenience. Coming back to the definitions, the above Fourier coefficients rewrite

$$\int_{[N_2]} \phi_{g,(0,1)}(uh)\psi^{-1}(u)\mathrm{d}u = \int_{[N_2]} \phi_{(0,1)}\left(\begin{pmatrix} uh & \\ & 1 \end{pmatrix} g\right)\psi^{-1}(u)\mathrm{d}u.$$

By writing these $\phi_{(0,1)}$ coefficients also explicitly, we get

$$\int_{[N_2]} \left[\int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} I & y \\ & 1 \end{pmatrix}\begin{pmatrix} uh & \\ & 1 \end{pmatrix} g\right)\psi^{-1}((0,1)y)\mathrm{d}y\right]\psi^{-1}(u)\mathrm{d}u. \tag{27}$$

Notice that

$$\begin{pmatrix} I & y \\ & 1 \end{pmatrix}\begin{pmatrix} uh & \\ & 1 \end{pmatrix} = \begin{pmatrix} u & y \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & 1 \end{pmatrix} \tag{28}$$

so that the above rewrites

$$\int_{[N_2]}\int_{(k\backslash\mathbf{A})^2} \phi\left(\begin{pmatrix} u & y \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & 1 \end{pmatrix} g\right)\psi^{-1}\begin{pmatrix} u & y \\ & 1 \end{pmatrix}\mathrm{d}y\mathrm{d}v \tag{29}$$

for the character defined by $\psi\left(\begin{smallmatrix} u & y \\ & 1 \end{smallmatrix}\right) = \psi(u)\psi(y)$. These matrices of the form $\left(\begin{smallmatrix} u & y \\ & 1 \end{smallmatrix}\right)$ describe the maximal unipotent subgroup N_3, so that we get the transfer formula

$$\int_{[N_2]} \phi_{g,(0,1)}(uh)\psi^{-1}(u)\mathrm{d}u = \int_{[N_3]} \phi\left(u \begin{pmatrix} h & \\ & 1 \end{pmatrix} g\right) \psi^{-1}(u)\mathrm{d}u. \tag{30}$$

The point of this rewriting is to be able to see the integrals over GL_2 (coming from the GL_2 Fourier expansion) as integrals over GL_3.

Remark. I believe this kind transfer formula is of fundamental importance in the works of JPSS, or the recent works of Humphries.

2.6. **Definitive Fourier expansion.** Recalling that $h = \left(\begin{smallmatrix} \delta & \\ & 1 \end{smallmatrix}\right)\gamma$ in the above. Putting back together what we worked out in this section, we obtain

$$\phi(g) = \sum_{\gamma \in P_2(k)\backslash \mathrm{GL}_2(k)} \sum_{\delta \in N_1(k)\backslash \mathrm{GL}_1(k)} \int_{N_3(k)\backslash N_3(\mathbf{A})} \phi\left(u \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi^{-1}(u)\mathrm{d}u.$$

Notice that $N_{n-2}(k)\backslash \mathrm{GL}_{n-2}(k) \simeq N_{n-1}(k)\backslash P_{n-1}(k)$: the subgroup $N_{n-1}(k)$ operates on the lines of matrices in $P_{n-1}(k)$, and we can make the last column of such matrix by these operations equal to $(0, \dots, 0, 1)$. Hence any element of $N_{n-1}(k)\backslash P_{n-1}(k)$ has a representant of the form $\left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$ with $A \in \mathrm{GL}_2(k)$, well defined modulo $N_{n-2}(k)$.

With these interpretation, we obtain

$$\begin{aligned}\phi(g) &= \sum_{\gamma \in P_2(k)\backslash \mathrm{GL}_2(k)} \sum_{\delta \in N_2(k)\backslash P_2(k)} \int_{N_3(k)\backslash N_3(\mathbf{A})} \phi\left(u \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi^{-1}(u)\mathrm{d}u \\ &= \sum_{\gamma \in N_2(k)\backslash \mathrm{GL}_2(k)} \int_{N_3(k)\backslash N_3(\mathbf{A})} \phi\left(u \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \psi^{-1}(u)\mathrm{d}u.\end{aligned}$$

Denoting by $W_\phi\left(\left(\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix}\right) g\right)$ the last integral, we obtain the Fourier expansion

$$\phi(g) = \sum_{\gamma \in N_2(k)\backslash \mathrm{GL}_2(k)} W_\phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right). \tag{31}$$

2.7. **And for** $\mathrm{GL}(n)$**?** By induction, we would prove exactly in the same manner that a cuspidal automorphic form ϕ on $\mathrm{GL}(n)$ admits the Fourier expansion

$$\phi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \tag{32}$$

where its Fourier coefficients are defined by

$$W_\phi(g) = \int_{N_n(k)\backslash N_n(\mathbf{A})} \phi(ug)\psi^{-1}(u)\mathrm{d}u. \tag{33}$$

3. Whittaker models and functionals

3.1. **Whittaker models.** It happens that the above Fourier coefficients W_ϕ have good properties:

- smooth on $G(\mathbf{A})$
- for $n \in N(\mathbf{A})$, we have $W(ng) = \psi(n)W(g)$
- moderate growth at $v \mid \infty$, i.e. $W\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) g\right) \ll |y|^A$, since ψ is of moderate growth.

Define the space of such functions

$$\mathcal{W}(\pi, \psi) = \{W_\phi \ : \ \phi \in V_\pi\} \tag{34}$$

the *Whittaker model* of π. The group $G(\mathbf{A})$ acts on $\mathcal{W}(\pi, \psi)$ by right translations, i.e. $g \cdot W_\phi(x) = W_\phi(xg)$.

Fact. *The map $\psi \mapsto W_\phi$ intertwines $V_\pi \to \mathcal{W}(\pi, \psi)$ i.e. $\mathcal{W}(\pi, \psi)$ is indeed a model (i.e. a realization) of V_π.*

Proof. The map is indeed G-linear, since

$$h \cdot W_\phi(g) = W_\phi(gh) = \int_{k\backslash \mathbf{A}} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} gh \right) \psi^{-1}(x)\mathrm{d}x = W_{h\cdot\phi}(g). \tag{35}$$

The map is also surjective by definition of $\mathcal{W}(\pi, \psi)$. It is injective by the above Fourier expansion: we can recover ϕ from the knowledge of its Fourier coefficient W_ϕ. □

Remark. This is not as proper as it should be: π is not a genuine representation of $G(\mathbf{A})$, so that the action has to be specified on the Whittaker functions too. This may be done for instance by saying that $(\mathfrak{g}_\infty, K_\infty)$-modules are equivalent to $\mathcal{H}$-modules where $\mathcal{H}$ is the global Hecke algebra, see e.g. Bump).

Define now a Whittaker model as being a *model* made of functions such that

$$W(ng) = \psi(n)W(g), \quad \text{for all } n \in N(\mathbf{A}).$$

3.2. **Whitatker functionals.** Consider $\Lambda(\xi) = W_\xi(1)$. It is a smooth function $V_\pi \to \mathbf{C}$ satisfying the condition $\Lambda(\pi(n)\xi) = \psi(n)\Lambda(\xi)$ for all $n \in N(\mathbf{A})$. Such functions are called *Whittaker functionals.*

Fact. *Having a Whittaker model is equivalent to having a Whittaker functional.*

Proof. We have

$$\begin{array}{ccc} WM & \longleftrightarrow & WF \\ (\xi \mapsto W_\xi) & \longmapsto & (\xi \mapsto W_\xi(1)) \\ (\xi \mapsto (g \mapsto \Lambda(\pi(g)\xi)) & \longleftarrow & \Lambda \end{array} \tag{36}$$

and we indeed have $W_\xi(ng) = \psi(n)W_\xi(g)$ and $\Lambda(n\xi) = \psi(n)\Lambda(\xi)$. □

Note that this equivalence between both languages is independent of the precise setting, global or local. The same definitions make sense in the local setting: Whittaker models and Whittaker functionals.

Remark. The above result is a way of stating Frobenius reciprocity: functions in $\mathcal{W}$ are such that $f(ng) = \psi(n)f(g)$ i.e. are in $\mathrm{Ind}_N^G(\psi)$. The reciprocity then is

$$\mathrm{Hom}_G(V, \mathrm{Ind}_N^G(\psi)) \simeq \mathrm{Hom}_N(V, \psi), \tag{37}$$

where the left factor is indeed the space of Whittaker models $\phi \mapsto W_\phi$ and the right factor is the space of Whittaker functionals $\phi \mapsto \Lambda_\phi$.

3.3. **Generic representations.**

Definition. We say that (π, V) is (ψ-)*generic* if it has a nonzero Whittaker model (or, equivalently, Whittaker functional) for ψ.

We proved above, by establishing its Fourier expansion, that a *cuspidal* representation do have Whittaker model, i.e. is generic (no matter the character ψ).

4. Local and global setting

4.1. **Local uniqueness.** A fundamental result in the theory of automorphic forms is:

Theorem (Local uniqueness)**.** *Let (π_v, V_{π_v}) irreducible admissible smooth representation of $G(k_v)$. The space of continuous Whittaker functionals is at most one-dimensional.*

For a proof, see e.g. Godement's notes on Jacquet-Langlands.

4.2. **Local-global interplay.** Let (π, V) be a smooth admissible representation of $G(\mathbf{A})$. By the Flath/tensor product theorem, $\pi \simeq \otimes'_v \pi_v$ and $V_\pi \simeq \otimes' V_{\pi_v}$.

Fact. *A global Whittaker functional of π corresponds to "suitable" local Whittaker functional families of the π_v.*

Proof. A Whittaker functional on π induces a Whittaker functional on the π_v by

$$\Lambda_v : V_{\pi_v} \hookrightarrow \otimes'_v V_{\pi_v} \xrightarrow{\sim} V_\pi \xrightarrow{\Lambda} \mathbf{C} \tag{38}$$

and $\Lambda = \otimes_v \Lambda_v$.

Conversely, a family of "suitable" local Whittaker functionals $(\Lambda_v)_v$, more precisely such that $\Lambda_v(\xi_v^\circ) = 1$ for K_v-fixed vectors at almost every place v (these ξ_v° exist a.e. since π_v is unramified a.e.), determines a global $\Lambda_v = \otimes'_v \Lambda_v$. □

Theorem. *(Global uniqueness) Let (π, V) admissible smooth irreducible representation of $G(\mathbf{A})$. The space of Whittaker functionals of π is at most one-dimensional.*

Proof. This is a corollary of the above comments: global Whittaker functionals correspond to families of (certain) local Whittaker functions, and these are unique. □

Corollary. *For a cuspidal representation π, globally generic implies locally generic.*

Proof. By cuspidality, there is a Whittaker functional. We can hence deduce local Whittaker functionals from a global Whittaker functional as in (38). □

Conjecture. (Shahidi?) If $\otimes_v \psi_v$ is a character of $k \backslash \mathbf{A}$, then locally generic implies globally generic.

4.3. **Factorization.** Uniqueness implies the following impressive result.

Corollary (Factorization of Whittaker models)**.** *Let (π, V) be a cuspidal representation, and $\pi \simeq \otimes'_v \pi_v$. Let $\phi \in V_\pi$ corresponding to $\otimes'_v \xi_v$ under the isomorphism. Then*

$$W_\phi(g) = \prod_v W_{\xi_v}(g_v). \tag{39}$$

Remark. This is *awesome*! Indeed, a priori a $\phi \in V_\pi$ (even cuspidal) does not factorize, but only does *under isomorphism*, which is totally unusable in practice. But its Whittaker model (which is unique, and contains all the information about π) does factorize!

Proof. Indeed we have

$$W_\phi(g) = \Lambda(\pi(g)\phi) = (\otimes'_v \Lambda_v)(\otimes'_v \pi_v(g_v)\xi_v) = \prod_v \Lambda_v(\pi_v(g_v)\xi_v) = \pi_v W_{\xi_v}(g_v). \quad \square \tag{40}$$

Remark. A clean proof of this fact, justifying all the embeddings, the well-definiteness of the restricted tensor products, and using induction on the set of places where ξ_v is not the "trivial vector" (with respect to which the restricted tensor product is defined) is available in Bump.

5. Multiplicity one

Theorem (Multiplicity one)**.** *Let* (π, V) *be a smooth, admissible, irreducible representation of* $\mathrm{GL}_n(\mathbf{A})$*. Then its multiplicity* $m(\pi)$ *in* $\mathcal{A}_{\mathrm{cusp}}$ *is at most one.*

Proof. Suppose that π is cuspidal, so that $m(\pi) \neq 0$. Take two copies of π in $\mathcal{A}_0^\infty$, say $\Phi_i : V_\pi \simeq V_{\pi,i} \subset \mathcal{A}_0^\infty$. For $\xi \in V_\pi$, let $\psi_i = \Phi_i(\xi)$ its image in $V_{\pi,i}$. Then

$$\xi \longmapsto \phi_i \longmapsto W_{\phi_i}(1) =: \Lambda_i(\xi) \tag{41}$$

gives two nonzero Whittaker functionals on V_π. By uniqueness, $\Lambda_1 = c\Lambda_2$. Thus

$$W_{\phi_1}(g) = \Lambda_1(\pi(g)\xi) = c\Lambda_2(\pi(g)\xi) = W_{\phi_2}(g) \tag{42}$$

by definition of Λ_i. Thus, using the Fourier expansion, we get

$$\phi_1(g) = \sum_{\gamma \in k^\times} W_{\phi_1}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) = c \sum_{\gamma \in k^\times} W_{\phi_2}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) = c\phi_2(g). \tag{43}$$

Thus, $V_{\pi,1} \cap V_{\pi,2} \neq 0$. By irreducibility, we deduce $V_{\pi,1} = V_{\pi,2}$. □

Theorem (Strong multiplicity one)**.** *Let* (π_1, V_1) *and* (π_2, V_2) *cuspidal representations on* $\mathrm{GL}_n(\mathbf{A})$*. Suppose* $\pi_{1,v} \simeq \pi_{2,v}$ *for all but a finite number of places. Then*

$$(\pi_1, V_1) = (\pi_2, V_2).$$

In other words, we can forget a finite number of places, and multiplicity one still holds.

6. Connection with L-functions (next talk!)

By the classical theory, for $f \in S_k(N, \psi)$, the associated L-function may be expressed as the "vertical" Mellin transform

$$L(s, f) = \int_0^\infty f(iy)|y|^s \mathrm{d}^\times y = \mathcal{M}_{i\mathbf{R}} f(s). \tag{44}$$

Adelizing as usual $f \mapsto \phi_f$, we have $f(iy) = \phi_f\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right)$, thus

$$L(s, f) = \int_{k^\times \backslash \mathbf{A}^\times} \phi\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right) |y|^s \mathrm{d}^\times y \tag{45}$$

and by Fourier expansion, this rewrites

$$L(s, f) = \int_{\mathbf{A}^\times} W_\phi\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right) |y|^s \mathrm{d}^\times y. \tag{46}$$

This opens the path to defining and studying L-functions via suitable integral representations built from the associated Whittaker models (the so-called Gelbart-Jacquet theory of zeta integrals).

7. Openings

Some comments towards other landscapes:

- Part of the theory extends to other groups, but then the genericity does depend on ψ. We still have multiplicity one, but cuspidal automorphic representations does not necessarily admit a Whittaker model (e.g. Siegel modular forms on $\mathrm{Sp}(2n)$.
- We can compute explicitly (i.e. give formulas) for Whittaker functions, as Shintani and Miyauchi did for $\mathrm{GL}(n)$. In general, there is the Casselman-Shalika formula.

- The space $\mathcal{W}(\pi, \psi)$ is made of nice functions, smooth and of moderate growth. However, it may be needed to have more precise analytic information, e.g. asymptotics (see Edgar Assing's talk).

References

[1] Bump (1998) *Automorphic forms and representations*, Cambridge University Press.
[2] Cogdell (2004) in *Lectures on automorphic L-functions*, Fields Institute, Amer. Math. Soc.
[3] Getz and Hahn (2022) *An introduction to automorphic representations*, GTM, Spinger.
[4] Godement (1970) *Notes on Jacquet-Langlands' theory*, Institute for Advanced Studies.
[5] Goldfeld and Hundley (2011) *Automorphic representations and L-functions for the General Linear Group*, volume II, Cambridge University Press.