LECTURE 7 — ARCHIMEDEAN ASPECT AND $(\mathfrak{g}, K)$-MODULES

TALK BY NICOLE RAULF

Abstract. In this talk we will consider the relation between representation theory and the classical theory of automorphic forms. We will review the basic ideas of Lie theory needed to treat the spectral decomposition in the cocompact case. If time permits we will also introduce $(\mathfrak{g}, K)$-modules to avoid analytic problems.

Let $(\pi, \mathfrak{h})$ a representation of the group G, that is to say

$$\pi : G \longrightarrow \mathrm{End}(\mathfrak{h}) \tag{1}$$

and so that the map $(g, f) \in G \times \mathfrak{h} \mapsto \pi(g)f$ is continuous. If $\mathfrak{h}$ is a Hilbert space and $\pi(g)$ is a unitary operator for all $g \in G$, the representation is called *unitary*.

This suffices to study $L^2(\Gamma\backslash G, \chi)$ (basically the space of automorphic forms) in the compact case, i.e. when $\Gamma\backslash G$ is compact. However, in the noncompact case there is a continuous part of the spectrum and there are important non-unitary representations (coming from the Eisenstein series). This is where the $(\mathfrak{g}, K)$-modules appear. The algebraic point of view of this theory has been developped by Harish-Chandra: the spirit is to mimic what we are able to do in the compact setting, typically by restricting to compact subgroups.

From now on let $G = \mathrm{GL}_n(\mathbf{R})$ or $\mathrm{GL}_n(\mathbf{R})^+$. Let $(\pi, \mathfrak{h})$ a representation of G on the Hilbert space $\mathfrak{h}$. We would like to get from it a representation of $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{R})$, which is not obvious if $\mathfrak{h}$ is infinite-dimensional.

Let $X \in \mathfrak{g}$. We want to define what $Xf := \pi(X)f$. These X will be seen as a differential operator. We let

$$Xf = \frac{\mathrm{d}}{\mathrm{d}t}\left(\pi(\exp(tX))\right) f_{|t=0} = \lim_{t\to 0} \frac{\pi(\exp(tX))f - f}{t} \tag{2}$$

if this limit exists.

We say that $f \in \mathfrak{h}$ is C^1 is the limit exists. Then $g \mapsto \pi(g)Xf$ is automatically continuous. Functions in the C^k class are defined as usual, inductively. Define $\mathfrak{h}^\infty$ be the space of smooth vectors, i.e. the subspace of C^∞ functions of $\mathfrak{h}$. This defines a Lie algebra representation of $\mathfrak{g}$ on $\mathfrak{h}^\infty$, that is we have the relation

$$[X, Y]f = X(Yf) - Y(Xf) \tag{3}$$

for all $X, Y \in \mathfrak{g}$. In other words, the representation $\rho : X \in \mathfrak{g} \mapsto X\cdot \in \mathrm{GL}(\mathfrak{h})$ satisfies

$$\rho([X, Y]_{\mathfrak{g}}) = \rho(X)\rho(Y) - \rho(Y)\rho(X) = [\rho(X), \rho(Y)]_{\mathrm{GL}}. \tag{4}$$

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These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

The space $\mathfrak{h}^\infty$ is invariant under the action of G. Then we automatically have an action of the universal enveloping Lie algebra $\mathcal{U}(\mathfrak{g})$ on $\mathfrak{h}^\infty$. This algebra $U(\mathfrak{g})$ can be thought of as the algebra of all differential operators, while $\mathfrak{g}$ only contains first order differential operators.

We now specialize on the right-regular representation of G. For f defined on G, for $X \in \mathfrak{g}$, define

$$(dXf)(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(g\exp(tX))_{|t=0} \tag{5}$$

the "partial derivative in the direction of X". Then f is smooth if and only if $dX_1 \circ dX_2 \circ \cdots \circ dX_n f$ exists and is continuous for all X_i's. Restricting from now on to smooth vectors is not totally crazy: $\mathfrak{h}^\infty$ is dense in $\mathfrak{h}$.

Let's decompose $(\pi, \mathfrak{h})$ is we restrict to a *maximal* compact subgroup K of G. Then (by averaging over K) there is a Hermitian inner product on $\mathfrak{h}$ such that

$$\langle \pi(K)u, \pi(K)v\rangle = \langle u, v\rangle, \quad \forall u, v \in \mathfrak{h}. \tag{6}$$

Let $(\pi_1, \mathfrak{h}_1)$ and $(\pi_2, \mathfrak{h}_2)$ be two representations of G. We say that $L : \mathfrak{h}_1 \to \mathfrak{h}_2$ is an *intertwining operator* if L is a linear continuous map and

$$\pi_2(g) \circ L = L \circ \pi_1(g), \quad \forall g \in G. \tag{7}$$

A *matrix coefficient* of a representation $(\pi, \mathfrak{h})$ of G is a function of the form $\mathfrak{g} \mapsto \langle \pi(g)x, y\rangle$ for some choice of $x, y \in \mathfrak{h}$.

The relation between reprensetations of G and K is given by the Peter-Weyl theorem.

Theorem. (Peter-Weyl) Let K be a compact subgroup of $\mathrm{GL}_n(\mathbf{C})$.

- The matrix coefficients of finite-dimensional unitary representations of K are dense in $\mathcal{C}(K)$ (space of continuous functions endowed with the L^∞-topology) and in all the $L^p(K)$ for indices $1 \leqslant p < \infty$.
- Any irreducible representation of K is finite-dimensional.
- If $(\pi, \mathfrak{h})$ is a unitary representation of K, then $\mathfrak{h}$ decomposes into a direct sum of irreducible unitary representations.

A representation $(\pi, \mathfrak{h})$ is called *admissible* if each isomorphism class of finite-dimensional unitary representation of K appears only finitely many times in π. This is essentially a finite-dimensional assumption (finitely-many pieces of inite dimension appearing in π). The admissible representations are the ones of interest to us (and the study of representations essentially boils down to it, by e.g. Langlands classification). From now on, all representations considered will be representations on G that are irreducible and K-admissible.

Let $\mathfrak{h}^K$ be the space of K-fixed vectors in $\mathfrak{h}$, i.e.

$$\mathfrak{h}^K = \{h \in \mathfrak{h} \ : \ \pi(k)v = v, \ \forall v \in \mathfrak{h}\}. \tag{8}$$

It is closed and invariant under $C_c^\infty(K\backslash G/K)$. The action is defined by averaging the action of G, weighted by f:

$$\pi(f)v = \int_G f(g)\pi(g)\mathrm{d}g. \tag{9}$$

By admissibility, $\mathfrak{h}^K$ is a finite-dimensional vector space. In fact we can show that the dimension is at most one.

Let $(\pi, \mathfrak{h})$ be an admissible representation of G. Assume that the restriction of π to K is unitary. Let σ be an isomorphism class of irreducible representation of K, and let $\mathfrak{h}(\sigma)$ be the sum of all irreducible invariant subspaces of $\mathfrak{h}$ that are isomorphic to σ. Then we have the "K-isotypic decomposition"

$$\mathfrak{h} = \widehat{\bigoplus}_{\sigma \in \hat{K}} \mathfrak{h}(\sigma) \tag{10}$$

where the direct sum is a Hilbert direct sum. Since π is admissible, the dimension of each $\mathfrak{h}(\sigma)$ is finite. Let $\mathfrak{h}_{\text{fin}}$ be those vectors whose projections to only a finite number of $\mathfrak{h}(\sigma)$ is nonzero (i.e. the algebraic direct sum). The elements of $\mathfrak{h}_{\text{fin}}$ are called *K-finite*.

Introduce $\mathfrak{k}$ the Lie algebra of K (skew-symmetric matrices). There is an equivalence between, for $f \in \mathfrak{h}$,

- f is K-finite
- $\langle \pi(k)f \ : \ k \in K \rangle$ is finite-dimensional
- $\langle Xf \ : \ X \in \mathfrak{k} \rangle$ is finite-dimensional

The K-finite vectors are smooth, and $\mathfrak{h}_{\text{fin}}$ is dense in $\mathfrak{h}^\infty$ (and thus in $\mathfrak{h}$) and invariant under the action of $\mathfrak{g}$ on $\mathfrak{h}^\infty$.

$(\mathfrak{g}, K)$-modules

We will look at the definition in the case of $G = \mathrm{GL}_n(\mathbf{R})$ (and $K = \mathrm{O}_n$) or $G = \mathrm{GL}_n(\mathbf{R})^+$ (and $K = \mathrm{SO}_n$). A $(\mathfrak{g}, K)$-module is a vector space endowed with a representation (π, V) of K and $\mathfrak{g}$ such that

- V decomposes into an algebraic direct sum of finite-dimensional K-invariant subspaces
- the actions of $\mathfrak{g}$ and K are compatible:
$$Xf = \lim_{t \to 0} \frac{1}{t} \left(\pi(\exp(tX))f - f \right) \tag{11}$$
for $f \in V$ and $X \in \mathfrak{k}$
- $\pi(g)\pi(X)\pi(g^{-1})f = \pi(\mathrm{Ad}(g)X)f$ for all $g \in K$ and $X \in \mathfrak{k}$.

Moreover, we say that V is K-finite if no K-invariant subspace arise with infinite multiplicity.

$(\mathfrak{g}, K)$-modules for $\mathrm{GL}(2, \mathbf{R})$

The definition is a special case of the above. From now on, let $G = \mathrm{GL}(2, \mathbf{R})^+$ and $K = \mathrm{SO}(2)$. Let V be a $(\mathfrak{g}, K)$-module, denote the action ok K on V by π as usual. We know that we can decompose

$$V = \bigoplus_{\sigma} V(\sigma) \tag{12}$$

where $V(\sigma)$ is the σ-isotypic component of V. Since V is a $(\mathfrak{g}, K)$-module, the $V(\sigma)$ are finite-dimensional.

Extend the action on $\mathfrak{g}$ to an action of the envelopping algebra $\mathcal{U}(\mathfrak{g})$. Since V is a complex vector space, calculations are done in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$. Consider V is an irreducible admissible $(\mathfrak{g}, K)$-module for G and $D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}_{\mathbf{C}}))$ (a differential operator) in the center of the envelopping complexified algebra. Then there exists a constant λ such that

$$Dx = \lambda x, \quad \forall x \in V. \tag{13}$$

Indeed, the commuting action of Z on each $V(\sigma)$ leads to a λ_σ and the above equation only on V_σ, and by irreducibility of V these λ_σ are all the same, so that the equation holds on all V.

In this situation G/K has dimension 2 so that there are two differential operators as expected: ∂_x and ∂_y. We have $K \simeq \mathbf{R}/\mathbf{Z}$ so that irreducible representations of K are given by $\sigma_k(K_\theta) := e^{ik\theta}$, where K_θ is the usual rotation matrix of angle θ. Then we can simply write

$$V = \bigoplus_k V(k) \tag{14}$$

where $V(k)$ is the σ_k-type.

Let Σ be the set of integers k such that $V(k) \neq 0$, i.e. the K-types (genuinely occurring in V).

$V(k)$ is the space of all $x \in V$ that have eigenvalue k, i.e. $Hx = kx$, where $H = -i \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This can be written down explicitly, cf. Bump. For $x \in V(k)$, there are elements $Rx \in V(k+2)$ (level raising operator) and $Lx \in V(k-2)$ (level lowering operator). The operators R and L are element of the Lie algebra. More explicitly, they are given by

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad \text{and} \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{15}$$

These operators can be written as classical differential operators in terms of ∂_x and ∂_y. Note that it is known that the center is always described as a polynomial algebra in some elements. In this case of GL(2), there is only one (the Casimir, which is the usual Laplacian), but for GL(3) there are already two.

Let X be a nonzero element in $V(k)$, then $\mathbf{C}x = V(k)$ and $\mathbf{C}R^n x = V(k+2n)$ and $\mathbf{C}L^n x = V(k-2n)$. We deduce the decomposition

$$V = \mathbf{C}x \oplus \bigoplus_{n>0} \mathbf{C}R^n x \oplus \bigoplus_{n>0} \mathbf{C}L^n x. \tag{16}$$

Indeed, since $\mathfrak{g}$ is generated by H, L, R, the above sum is the $\mathfrak{g}$-module generated by x, which by irreducibility has to be the whole V. We moreover have that $\dim V(k) \leqslant 1$, and all the K-types k have same parity.

We are now aiming at giving a description of all the $(\mathfrak{g}, K)$-modules in this case. Define the Laplace operator by

$$-4\Delta = H^2 - 22RL + 2RL. \tag{17}$$

If λ is an eigenvalue of Δ on V, x an eigenfunction and $x \in V(k)$, then

$$LRx = (-\lambda - \frac{k}{2}(1 + \frac{k}{2}))x \tag{18}$$

since such a scalar exists by one-dimensionality, and then can be computed. We have a similar formula for RLx.

If a $(\mathfrak{g}, K)$-module exists, then it is unique. Let $k \geqslant 1$ and λ of the form $\frac{k}{2}(1 - \frac{k}{2})$ and V irreducible admissible $(\mathfrak{g}, K)$-module with the same parity as k. Then the set of integers is among the three following families (maybe only when λ is of this form above, otherwise it corresponds to Maass forms, see Bump)

- $\Sigma^+(k) = \{\ell \in \mathbf{Z} \ : \ l \equiv k \mod 2, \ell \geqslant k\}$
- $\Sigma^-(k) = \{\ell \in \mathbf{Z} \ : \ l \equiv k \mod 2, \ell \leqslant -k\}$
- $\Sigma^0(k) = \{\ell \in \mathbf{Z} \ : -k < \ell < k\}$

Then we can show that these objects do exist (either by making them explicit, or by considering the representation theory of GL(2)), so we have described them all.

[Do a drawing of the series.]