LECTURE 6 — WHITTAKER MODELS. AN OVERVIEW

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Abstract. We will define the notion of a Whittaker model in the local setting and discuss the global Whittaker period of an automorphic form. To provide some practical context to these (technical) notions we will sketch how to derive the Whittaker expansion for some (generic) automorphic forms. Finally we will indicate how this expansion can be useful for certain estimates.

We will introduce Whittaker models.

Consider F a local (e.g. $\mathbf{Q}_p$ or $\mathbf{R}$) or global field (e.g. $\mathbf{Q}$). Let ψ_F an additive character on F, in the global case $\psi_F : F\backslash\mathbf{A} \to S^1$. Consider $G = \mathrm{GL}(n)$, even though most of the definitions work for more general reductive quasi-split groups. Let N be the standard upper unipotent subgroup, and for $n \in N$ let $\psi_N(n) = \psi_F(t_1x_1 + \cdots + t_{n-1}x_{n-1})$ for coefficients $t_i \in F$. All the additive characters of N are of this form (and they are nondegenerate if $t_i \in F^\times$. In general, we can define such characters from the simple roots of the quasi-split reductive group G.

1. Local setting

Let F be a local field. Let $\mathcal{W}(\psi_N) = \mathrm{Ind}_N^G(\psi_N)^\infty$ the space of smooth vectors in the induced representation. It is

$$\{W : G \to \mathbf{C} \ : \ \text{smooth and } \forall g \in G(F); \forall n \in N(F), W(ng) = \psi_N(n)W(g)\} \tag{1}$$

and G acts on $\mathcal{W}(\psi_n)$ by right regular representation.

Fact. $\mathcal{W}(\psi_N)$ is multiplicity free.

Proof. The proof is pretty difficult but important (see Shalika [?] for archimedean fields, and Gelfand-Khazdhan for non-archimedean ones). ☐

Definition. A smooth irreducible representation (π, V_π) of $G(F)$ is called *generic* if

$$\mathrm{Hom}_G(V_\pi, \mathcal{W}(\psi_N)) \neq \{0\}. \tag{2}$$

If π is generic, we have an intertwining injecting

$$\begin{array}{ccc} V_\pi & \longrightarrow & \mathcal{W}(\psi_N) \\ v & \longmapsto & W_v \end{array} \tag{3}$$

The image is written $\mathcal{W}(\pi, \psi_N)$ and is called the ψ_N-Whittaker model of π.

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These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

Remark. If there is a ψ_N-Whittaker model, then there is also a ψ'_N-Whittaker model for any ψ'_N. This is not true for e.g. the metaplectic double cover of SL(2), so that in general the notion of genericity does depend on the choice of additive character. But not for GL(n).

Note that π is generic if and only if there is a nontrivial continuous (we have to be careful in the case we are on $\mathbf{R}$) functional

$$\Lambda \quad V_\pi \quad \longrightarrow \quad \mathbf{C} \tag{4}$$

such that

$$\Lambda(\pi(n)v) = \psi_N(n)\Lambda(v). \tag{5}$$

Indeed, this is equivalent since if we have a Whittaker model W_v then we can define the Whittaker functional

$$\Lambda : v \mapsto W_v \mapsto W_v(1). \tag{6}$$

And conversely, if we have a Whittaker functional then we get the Whittaker model

$$\mathcal{W}(\pi, \Psi_N) = \{g \mapsto \Lambda\pi(g) \ : \ v \in V_\pi\}. \tag{7}$$

Fact. Suppose $F = \mathbf{Q}_p$ and v is $N(\mathbf{Z}_p)$-invariant (e.g. spherical vector) and $\psi_N(x) = \psi_F(x_1 + \cdots + x_{n-1})$, for ψ_F trivial on $\mathbf{Z}_p$ and nontrivial on $p^{-1}\mathbf{Z}_p$. Then

$$W_v\left(\begin{pmatrix} y_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}\right) = 0 \text{ for } y \in \mathbf{Q}_p \setminus \mathbf{Z}_p. \tag{8}$$

Proof. Let $y \in \mathbf{Q}_p^\times$. Consider $\bar{x} \in \mathbf{Z}_p$ and

$$X = \begin{pmatrix} 1 & \bar{x} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Then

$$W_v(a(y)) = W_v(a(y)X) = \psi_F(\bar{x}y)W_v(a(y)) \tag{10}$$

and $\psi_F(\bar{x}y) \neq 1$ for a certain $\bar{x}$ since ψ_F is not trivial. □

Fact. If π is generic, so is $\tilde{\pi}$.

Proof. Indeed, letting ω being the long Weyl element, the Whittaker model is in fact

$$\mathcal{W}(\tilde{\pi}, \psi_N^{-1}) = \left\{\tilde{W}(g) = W(\omega^t g^{-1}\right\}. \tag{11}$$

2. Global setting

Let $F = \mathbf{Q}$ for simplicity, but the general definitions carry on to any number field. Let (π, V_π) be a smooth cuspidal (irreducible) automorphic representations. Let a character $\psi : \mathbf{Q}\backslash\mathbf{A} \to S^1$, splitting into $\psi = \otimes_v \psi_v$.

We call π globally generic if the Fourier-Whittaker coefficient (or the Jacquet period)

$$W_\phi(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(ng)\psi_N^{-1}(n)\mathrm{d}n \tag{12}$$

is nonzero, as function on G. We then obtain a Whittaker functional by considering

$$\begin{array}{rcl}\Lambda\colon V_\pi & \longrightarrow & \mathbf{C} \\ \phi & \longmapsto & W_\phi(1)\end{array} \tag{13}$$

As before, we deduce from it a global Whittaker model $\mathcal{W}(\pi, \psi_N)$. **Question.** Can we define globally generic as a global analogue of locally generic? Existence of a Whittaker model, of an interwiner?

We call $\pi \simeq \otimes_v \pi_v$ locally generic if, for all place v, there is an additive character ψ_v so that π_v is generic with respect to ψ_v (for $\mathrm{GL}(n)$ it does not matter, but the definition makes sense more generally).

Lemma. Globally generic implies locally generic.

Proof. Take $\psi = \psi_{\mathbf{Q}_v}$ and define the local Whittaker functional

$$\Lambda_v : V_{\pi_{v_0}} \to \bigotimes_v V_{\pi_v} \simeq V_\pi \to \mathbf{C} \tag{14}$$

where the last arrow is given by Λ, the global Whittaker functional. □

Remarks. Some important comments:

- **Global uniqueness.** If π is any irreducible smooth representations of $G(\mathbf{A})$, then the space of Whittaker functionals is at most one-dimensional. (i.e. if it is generic, then it is exactly one-dimensional)
- **Factorization.** We have $\Lambda = \bigotimes_v \Lambda_v$. Indeed, for ϕ corresponding to $\otimes'_v \xi_v$ (through the isomorphism $\pi \simeq \otimes_v \pi_v$, but note it is not the $\phi \in V_\pi$ that factorizes since it is in a kind of rigid L^2 space, but only through an isomorphism) we have, for all $g \in G(\mathbf{A})$,

$$W_\phi(g) = \prod_v W_{\xi_v}(g_v). \tag{15}$$

3. Some conjectures

Take G a quasi-split group and $G = NAK$, with Borel $B = NA$.

Conjecture. Suppose π is a cuspidal ψ_v-locally generic for every place v. Assume $\otimes_v \psi_v = \psi$ is a character for $\mathbf{Q}\backslash\mathbf{A}$ (in general for random ψ_v's, it can not be $\mathbf{Q}$-invariant). Then π is ψ-globally generic.

Conjecture. Suppose π is a cuspidal locally generic representation. Then π is tempered. (For $\mathrm{GL}(n)$ this means that cusp forms are all tempered: it is the Ramanujan conjecture)

Conjecture. In every (local) tempered L-packet, there is a unique generic representation.

Precise statements and further discussion of these conjectures can be found in [**?**].

4. Fourier expansion

For $\mathrm{GL}(2)$, observe that $N(F) \simeq F$ is a nice abelian group. The Jacquet period is then just the Fourier transform. Classical Fourier analysis then yields

$$\phi(g) = \sum_{\gamma \in \mathbf{Q}^\times} W_\phi\left(\left(\begin{smallmatrix}\gamma & 0 \\ 0 & 1\end{smallmatrix}\right) g\right) \tag{16}$$

for $\phi \in V_\pi$ and π a cuspidal representation. Using support properties as in (**??**) one sees that for any $g \in G(\mathbb{A})$ and any ϕ, there is $K = K(g, \phi)$ so that the γ-sum is supported in $\frac{1}{K}\mathbf{Z} \setminus \{0\}$.

For $n \geqslant 3$ and $G = \mathrm{GL}(n)$, we still get a (Fourier-)Whittaker expansion for cuspidal ϕ:

$$\phi(g) = \sum_{\gamma \in N_{n-1}(\mathbf{Q}) \backslash \mathrm{GL}_{n-1}(\mathbf{Q})} W_\phi \left(\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix} \right) g \right). \tag{17}$$

For g upper triangular, we can conjugate and still get explicit expression.

Corollary. Each cuspidal automorphic representation of $\mathrm{GL}(n)$ is globally generic. (and in particular the first conjecture above is true in the case of $\mathrm{GL}(n)$).

Proof. To be written □

5. Some words on GSp(4)

The classical Siegel modular forms are not generic, so that things must work differently.

Let $G = \mathrm{GSp}(4)$. In this case, the N is rather different, it is the set of matrices of the form

$$n = \begin{pmatrix} 1 & x & a & b + xc \\ & 1 & b & d \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \tag{18}$$

Then a character is defined by $\psi_N(n) = \psi_F(x + d)$. Given a cuspform, we again have

$$W_\phi(g) = \int_{N(\mathbf{Q}) \backslash N(\mathbf{A})} \phi(ng) \psi_N(n)^{-1} \mathrm{d}n. \tag{19}$$

Remark. This has not to be mistaken with the Fourier theory for Siegel modular form, where we consider the abelian subgroup of N given by $x = 0$.

Consider the subgroup U of matrices

$$u = \begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{20}$$

Set

$$\phi^U(g) = \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \phi(ug) \mathrm{d}g. \tag{21}$$

Then we have a Whittaker expansion

$$\phi^U(g) = \sum_{x \in N_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{Q})} W_\phi \left(\begin{pmatrix} 1 & & & \\ & a & & b \\ & & \det(x) & \\ & c & & d \end{pmatrix} \right) \tag{22}$$

where x is the matrix (a, b, c, d) that we injected in $\mathrm{GSp}(4)$ this way. This expansion appears implicitly in the unfolding process for the Novordvorski integral representation of the spinor L-function. (The expansion is given explicitly with proof in some unpublished notes of B. Roberts and R. Schmidt.)

If π is non-generic, then $\pi^U(g) = 0$ for all g and all $\phi \in V_\pi$. This is called "hypercuspidality". The converse is immediate by Fubini (splitting $\mathrm{d}n$ by $\mathrm{d}u$ and making ϕ^U appear as a factor). Similarly (using Fubini) one sees that hypercuspidal implies cuspidal (one notes that U is in the intersection of all standard Parabolic subgroups).

Question. What is tempered? Globally tempered means locally tempered. And locally tempered can be phrased in terms of regularity of matrix coefficients.

References

[Sh] F. Shahidi, *Arthur packets and the Ramanujan conjecture*, Kyoto J. Math. 51 (2011).

[Sha] J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. (2) 100 (1974).