LECTURE 5 — KATZ' CONSTRUCTION OF THE EISENSTEIN MEASURE FROM AN AUTOMORPHIC VIEWPOINT

TALK BY ADEL BETINA

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Abstract. We will talk about Eisenstein series and measures [4, 1]. They are very important to study integrality of Hecke L-values, but often come with a heavy load of computations. Harris, Hsieh, Li et al [2, 3] began studying these Eisenstein series using automorphic forms instead, reducing the computations to simpler studies of zeta integrals, Mellin transform, etc. We will present how to estimate Eisenstein series for GL(2) at CM points with this approach.

Let F be a totally real field with ring of integers $\mathfrak{o}_F$. Denote by v a place of F, and by $\mathfrak{o}_v$ its ring of integers.

1. Eisenstein series for GL(2)

Denote $\Sigma_F = \mathrm{Hom}_{\mathrm{alg}}(F, \overline{\mathbf{Q}})$ the set of embeddings of F in $\overline{\mathbf{Q}}$. Let $\chi_1, \chi_2 : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ Hecke characters. Introduce the induced representation $I(\chi_1, \chi_2, s)$ defined by the space

$$\left\{ f : \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C} \ : \ \text{smooth, } K_\infty\text{-finite and } f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \right) = \chi_1(a)\chi_2(d) \left| \frac{a}{d} \right|^{s+1/2} f(g) \right\}.$$

Note that these functions are not automorphic, since we did not impose $\mathrm{GL}_2(F)$-invariance: it is only a certain induced representation.

Let's build a map $I(\chi_1, \chi_2, s) \to \mathcal{A}([\mathrm{GL}_2], \chi_1\chi_2)$ which is $\mathrm{GL}_2(\mathbf{A}_F)$-equivariant (by right translations). We make $\mathrm{GL}_2(F)$-invariant a function $f \in I(\chi_1, \chi_2, s)$ by essentially averaging over $\mathrm{GL}_2(F)$. Since the action of $B(F)$ on f is already determined by definition of $I(\chi_1, \chi_2, s)$, it is enough to average over $N(F)\backslash \mathrm{GL}_2(F)$. We hence define

$$f \longmapsto \left(g \mapsto E(g, s) := \sum_{\gamma \in B(F)\backslash \mathrm{GL}_2(F)} f(\gamma g) \right) \tag{1}$$

where the sum on the right converges for $\Re(s) \gg 0$ for nice enough functions f (ultimately we will have a functional equation and then we will be able to extend it meromorphically).

2. Fourier coefficients

Let the additive character $\psi_{\mathbf{Q}} : \mathbf{A}_{\mathbf{Q}}/\mathbf{Q} \to \mathbf{C}^\times$ such that, at the archimedean place, $\psi_{\mathbf{Q},\mathbf{R}}(x) = \exp(2i\pi x)$. Let $\psi_F = \mathbf{A}_F/F \to \mathbf{C}^\times$ defined by $\psi_{\mathbf{Q}} \circ \mathrm{tr}_{F/\mathbf{Q}}$.

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Now, consider $x \mapsto E\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g, f\right)$ with $x \in \mathbf{A}_F/F$. It is a continuous function, periodic, and hence has Fourier expansion of the form

$$E\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, f\right) = \sum_{\alpha \in F} c_\alpha(g, f)\psi(\alpha x) \tag{2}$$

where the Fourier coefficients are given by

$$c_\alpha(g, f) = \int_{\mathbf{A}_F/F} E\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} g, f\right) \psi(-\alpha y) \mathrm{d}y. \tag{3}$$

We would like to see it instead as an integral over $\mathbf{A}_F$, so that it can be decomposed as a product of local integrals.

We can rewrite, evaluating at $x = 0$,

$$E(g, f) = \sum_{\alpha \in F} c_\alpha(g, f). \tag{4}$$

Set

$$W(g, f) = \int_{\mathbf{A}_F} f\left(\omega_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) \mathrm{d}x, \tag{5}$$

where ω_0 is the long Weyl element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Introduce also the intertwinning operator

$$Mf(g) = \int_{\mathbf{A}_F} f\left(\omega_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x. \tag{6}$$

Proposition. We have

$$E(g, f) = f(g) + Mf(g) + \sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f\right). \tag{7}$$

Proof. We will use the Bruhat decomposition to understand better $B(F)\backslash \mathrm{GL}_2(F)$:

$$B(F)\backslash \mathrm{GL}_2(F) = \{I_2\} \sqcup \omega_0 N(F). \tag{8}$$

Let $\alpha \neq 0$. We then have,

$$c_\alpha(f, g) = \int_{\mathbf{A}_F/F} \sum_{\lambda \in F} f\left(\omega_0 \begin{pmatrix} 1 & x+\lambda \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha(x+\lambda)) \mathrm{d}x + \int_{\mathbf{A}_F/F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) \mathrm{d}x. \tag{9}$$

The second integral is zero, since the value of f is $f(g)$ by the definition of the induced representation $I(\chi_1, \chi_2, s)$, hence we get the integral of a nontrivial character, which is hence zero.

Concerning the first integral, we can collapse the sum and the integral in the domain $\mathbf{A}_F$ by definition. It becomes the desired term

$$\int_{\mathbf{A}_F} f\left(\omega_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\alpha x) \mathrm{d}x = W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f\right). \tag{10}$$

In the case $\alpha = 0$, we obtain an analogous computation but then the second integral does not vanish:

$$\int_{\mathbf{A}_F/F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x = f(g), \tag{11}$$

and the first one directly gives the intertwining operator. □

Now the magic will be that, up to choosing cleverly the functions we integrate, these integrals will be computable.

3. Godement sections

Set $\chi = \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ of type $k\Sigma_F$ with $k \in \mathbf{Z}_{\geq 2}$, i.e.

$$\chi(z) = \prod_{\sigma\in\Sigma_F} \sigma(z)^k. \tag{12}$$

Let $\phi = \otimes_v \phi_v \in S(\mathbf{A}^2)$ a Schwartz function, i.e. smooth at all places (i.e. locally constant at finite places, and smooth and rapidly decaying at archimedean places), and at almost every places $\phi_v = \mathbf{1}_{\mathfrak{o}_v\times\mathfrak{o}_v}$. Define

$$f_{\chi_v,\phi_v,s}(g_v) = |\det g_v|_v^s \int_{F_v} \phi_v((0,t_v)g_v)\chi_v(t_v)|t_v|_v^{2s}\mathrm{d}^\times t_v, \tag{13}$$

and take the global section as the product of local ones:

$$f_{\chi,\phi,s}(g) = \bigotimes_v f_{\chi_v,\phi_v,s}(g_v) \in I(1,\chi^{-1},s-\tfrac{1}{2}). \tag{14}$$

4. CM Hilbert-Blumenthal Abelian varieties (HBAV)

Let K/F be a quadratic CM extension and $\iota_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ a fixed embedding. We say that p is ordinary in K if each factor of p in F splits in K. Let $\Sigma \subset \mathrm{Hom}_{\mathrm{alg}}(K,\overline{\mathbf{Q}})$ such that

- $\Sigma \sqcup \Sigma^c = \mathrm{Hom}_{\mathrm{alg}}(K,\overline{\mathbf{Q}})$,
- Letting Σ_p be the places (the primes) singled out by $i_p \circ \Sigma$, then $\Sigma_p \cap \Sigma_p^c = \varnothing$.

In other words, we are choosing, above each place of F, one of the two possible places.

Let $\lambda : \mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ of type $k\Sigma + \nu(1-c)\Sigma$ with $k \in \mathbf{Z}_{\geq 1}$ and $v = \sum v_\sigma \sigma$ with each $v_\sigma \in \mathbf{Z}_{\geq 0}$. Let $\mathcal{C}_\lambda$ be the conductor of λ and N be the level (norm) of $\mathcal{C}_\lambda$. For simplicity, assume that p is prime to $\mathcal{C}_\lambda$.

Set $\mathfrak{c}$ be an ideal of F and $\mathrm{GL}_2(\widehat{F})^{(\mathfrak{c})}$ be the set of $g \in \mathrm{GL}_2(\widehat{F})$ such that $\det(g)\widehat{\mathfrak{o}}_F = \mathfrak{c}\widehat{\mathfrak{o}}_F$. Consider the Shimura variety

$$\mathrm{Sh}(\mathfrak{c}, U_N(p^n)) = \mathrm{SL}_2(F)\backslash\mathbb{H}_F \times \mathrm{GL}_2(\widehat{F})^{(\mathfrak{c})}/\Gamma(Np^n)\cap \mathrm{SL}_2(\widehat{F}). \tag{15}$$

Consider now $x = (z,g) \in \mathbb{H} \times \mathrm{GL}_2(\widehat{F})^{(\mathfrak{c})}$. Introduce the lattice

$$\mathbb{L} = \mathfrak{o}_F e_1 \oplus \mathfrak{d}_F^{-1} e_2 \subset F^2 = V = Fe_1 \oplus Fe_2 \tag{16}$$

where $\mathfrak{d}_F$ is the different, and

$$\mathbb{L}_g = (\mathbb{L}\otimes\widehat{\mathbf{Z}})g^{-1} \cap V. \tag{17}$$

We have

$$L_x = q_z(\mathbb{L}_g) \tag{18}$$

with $q_z(te_1 + ye_2) = tz + y$. We have $L_x \subset (F\otimes\mathbf{C})$ and $A_x = F\otimes\mathbf{C}/L_x$ is a HBAV.

We are ready to define the relevant test-functions in this setting. Pick $v \in K$ such that

- $\bar{v} = -v$ with $\Im\sigma(v) > 0$ for all $\sigma \in \Sigma$,
- $D_F^{-1}(2vD_{K/F}^{-1})$ prime to $pND_{K/F}^{-1}$.

Define $i : K \to M_2(F)$ the embedding in quaternionic algebra, defined by

$$av + b \longmapsto \begin{pmatrix} b & v^2 a \\ a & b \end{pmatrix} \tag{19}$$

and consider $i_\Sigma : K \to K \otimes \mathbf{R} \simeq F \otimes \mathbf{C} = \mathbf{C}^\Sigma, x \mapsto (\sigma(x))_{\sigma\in\Sigma}$. For any integral ideal $\mathfrak{a}$ of K, we let $x(\mathfrak{a})$ be the HBAV corresponding to $[(i_\Sigma(v), i(\widehat{\mathfrak{a}})h)]$, where $\widehat{\mathfrak{a}} \in \mathbf{A}_K^\times$ is the idele of $\mathfrak{c}$ and h satisfies

$$q_v((\mathbb{L} \otimes \widehat{\mathbf{Z}})h^{-1}) = \widehat{\mathfrak{o}}_K. \tag{20}$$

4.1. **Special Godement sections.** Now we will introduce the Bruhat-Schwartz function to get the desired Eisenstein series. Following Ming-Lun, let $\phi = \otimes_v \phi_v$ where

- if $v \mid \infty$, let
$$\phi_v = 2^{-k}(x+iy)^k \exp(-\pi(x^2+y^2)), \tag{21}$$
- if $v \nmid p\mathcal{C}_\lambda\mathcal{C}_\lambda^c$, let
$$\phi_v = \mathbf{1}_{\mathfrak{o}_v \times \mathfrak{d}_v^{-1}}, \tag{22}$$
- if $v \mid \mathcal{C}_{\lambda,\mathrm{inert}}$, let
$$\phi_v = \mathbf{1}_{\mathfrak{o}_{K_v}^\times}(xe_{1,v} + ye_{2,v})\lambda(xe_{1,v}+ye_{2,v})^{-1} \quad \text{where} \quad \mathfrak{o}_{K_v} = \mathfrak{o}_v e_{1,v} \oplus \mathfrak{d}_v^{-1} e_{2,v}, \tag{23}$$

  if $v \mid p\mathcal{C}_{\lambda,\mathrm{split}}$:
  $v\mathfrak{o}_K = w\bar{w}$ with $w \mid \Sigma_p \mathcal{C}_{\lambda,\mathrm{split}}$, then
$$\phi_v(x,y) = \varphi_{\bar{w}}(x)\widehat{\varphi}_w(y),$$
  where $\varphi_w(y) = \mathbf{1}_{\mathfrak{o}_v^\times}(y)\lambda_w(y)$ and
$$\varphi_{\bar{w}}(x) = \begin{cases} \mathbf{1}_{\mathfrak{o}_v}(x) & \text{if } \bar{w} \nmid \mathfrak{F}_c, \\ \mathbf{1}_{\mathfrak{o}_v^\times}(x)\lambda_{\bar{w}}^{-1}(x) & \text{if } \bar{w} \mid \mathfrak{F}_c, \end{cases}$$
  where $\mathcal{C}_{\lambda,\mathrm{split}} = \mathfrak{F}\mathfrak{F}_c$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$.

Theorem. [Ming-Lun [3]] Set $\lambda_+ = \lambda_{|\mathbf{A}_F^\times}$. Then $E(\cdot, f_{\lambda_+,\phi,s}) \mid_{s=0}$ defines a holomorphic Hilbert Eisenstein series of weight k and level $\Gamma(N)$ with coefficients in a number field L (the coefficients are p-adically integral and interpolable). Moreover, $E(\cdot, f_{\lambda_+,\phi,s}) \mid_{s=0}$ is a toric Eisenstein series of eigencharacter λ.

4.2. **Toric Eisenstein series.** We let θ^ν be the Maass–Shimura's differential operator of [3, §.4.5], then $E^{nh} = \theta^\nu(E(\cdot, f_{\lambda_+,\phi,s})_{|s=0})$ is a nearly holomorphic Eisenstein series. Ming-Lun introduced and computed the following Eisenstein period integral

$$\int_{K^\times \mathbf{A}_F^\times \backslash \mathbf{A}_K^\times} E^{nh}(i(z)h)\lambda(z)\overline{\mathrm{d}^\times z}. \tag{24}$$

The integrand is really invariant by $K^\times \mathbf{A}_F^\times$, thus we can reduce to an integral over $\mathbf{A}_K^\times$ (writing the integral as an explicit sum, and $f_{\lambda_+,v,\phi_v,s}$ is an integral over $F_v^\times$, and split as a product of local zeta integrals that are computables):

$$\begin{aligned}\sum_{[A]\in Cl(K)/Cl(F)} E^{nh}(x(A), f_{\lambda_+,\phi,s})_{|s=0}\lambda(A) &= \int_{K^\times \mathbf{A}_F^\times \backslash \mathbf{A}_K^\times} E^{nh}(i(z)h)\lambda(z)\overline{\mathrm{d}^\times z} \\ &= L(0,\lambda) \times \mathrm{Eul}_p(\lambda) \times \text{extra factors,}\end{aligned}$$

θ^ν corresponds to a certain weight raising differential operator acting on the archimedean components of $E(\cdot, f_{\lambda_+,\phi,s})$.

where $x(A)$ is the CM HBAV corresponding to $[(i_\Sigma(v), i(\widehat{A})h)]$ as in §.4.

The Fourier coefficients of $E^{nh}(x(A), f_{\lambda_+,\phi,s})_{|s=0}$ are a priori difficult, but in fact they are always of the form of products of $a + b\lambda(z)$ and we can interpolate p-adically them after taking the p-depletion to build the Katz p-adic L-function.

This method generalizes to higher rank groups, with adequate modification of the terms appearing in the integrals.

References

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