LECTURE 4 — SPECTRAL THEORY OF L^2 SPACES

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Abstract. We sketch the decomposition of the right-regular representation of $L^2(\mathrm{SL}(2,\mathbf{Z})\backslash\mathrm{SL}(2,\mathbf{R}))$. Doing so we introduce the theory of Eisenstein series with their meromorphic continuation and functional equation. Time permitting, we relate this decomposition to the spectral theory of the hyperbolic Laplacian, and make some remarks about how the decomposition generalises to congruence subgroups and then to higher rank groups like $\mathrm{SL}(3,\mathbf{R})$.

The purpose is to understand functions in L^2 spaces (typically *automorphic forms*). Let G be a topological group. On $L^2(G)$ there is a natural action, the right regular representation defined by

$$\begin{array}{rccc} \rho & : & G & \longrightarrow & U(L^2(G)) \\ & & f & \longmapsto & [f_g : x \mapsto f(xg)] \end{array} \tag{1}$$

The aim is to decompose, in some sense, this right regular representation.

1. First baby example of spectral theory : $\mathbf{R}/\mathbf{Z}$

The L^2 space on $\mathbf{R}/\mathbf{Z} = \mathbf{T}$ gives rise to the theory of *Fourier series*: any function $f \in L^2(\mathbf{T})$ can be expanded in the form

$$f(t) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)e(nt) \tag{2}$$

where $e(t) = \exp(2i\pi t)$ and, for $n \in \mathbf{Z}$,

$$\widehat{f}(n) = \int_{\mathbf{R}/\mathbf{Z}} f(t)e(-nt)\mathrm{d}t \tag{3}$$

is the nth Fourier coefficient of f.

The L^2 property is not necessary to have a Fourier series decomposition, it is endowed with the particular structure of an Hilbert space and we will restrict to this setting, natural ground for spectral theory.

The basis $(e(nt))_n$ is not just a random basis, but has the following important properties:

- $e(nt)$ is an eigenfunction of the Laplacian $\Delta_{\mathbf{T}} = \partial^2_{tt}$
- $\mathbf{C}e(nt)$ is $\mathbf{R}$-invariant under the right regular representation, i.e. is an irreducible subrepresentation of the right regular action

$$\begin{array}{rcc} \rho \ : \mathbf{R} & \longrightarrow & U(L^2(\mathbf{T})) \\ f & \longmapsto & f_u : t \mapsto f(t+u). \end{array} \tag{4}$$

- The right regular representation commutes with the Laplacian $\Delta_{\mathbf{T}}$.

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These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

2. Second baby example of spectral theory : R

The L^2 space on $\mathbf{R}$ gives rise to the theory of *Fourier transform*: any function $f \in L^2(\mathbf{R})$ can be expanded as

$$f(t) = \int_{\mathbf{R}} \widehat{f}(\xi)e(\xi t)\mathrm{d}\xi \tag{5}$$

where (in the case of a Schwartz function, then we extend the notion of Fourier transform by density, even though without the following formula), for $\xi \in \mathbf{R}$,

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(t)e(-\xi t)\mathrm{d}t \tag{6}$$

is the Fourier transform of f.

The spirit is the same as in the previous section, thinking of integrals as continuous analogues of sums. This can be seen as a *direct integral* decomposition in a specific sense. The "basis" $(e(\xi t))_\xi$ has the following important properties:

- $e(\xi t)$ is an eigenfunction of the Laplacian $\Delta_{\mathbf{R}} = \partial^2_{tt}$
- $\mathbf{C}e(nt)$ is $\mathbf{R}$-invariant under the right regular representation (we could see it as a "subrepresentation" of the right regular representation, even though it is not, see the next comment).

However, unlike the previous case, $e(\xi t)$ is not anymore in $L^2(\mathbf{R})$. But we still have a formal decomposition of $L^2(\mathbf{R})$. We can think of the Fourier transform as a projection

$$f \mapsto \widehat{f}(\xi)e(\xi t) \tag{7}$$

but this is defined only almost everywhere (we need the formalism of direct integral decomposition to state it properly). This "projection" is almost $\mathbf{R}$-equivariant, and every L^2 function can be reconstructed by these projections (via the Fourier inversion formula).

3. The case of SL(2, R)

Our aim is to treat the case $G = \mathrm{SL}(2, \mathbf{R})$ and $\Gamma = \mathrm{SL}(2, \mathbf{Z})$. We work with the space $L^2(\Gamma\backslash G)$ defined as function on $\Gamma\backslash G$ (i.e. functions on G that are Γ-invariant) such that $\|f\|_2$ is finite, for the norm induced by the inner product

$$\langle f, g\rangle = \int_G f(x)g(x)\mathrm{d}x \tag{8}$$

with $\mathrm{d}x$ a Haar measure on $\Gamma\backslash G$.

We saw in the baby cases that the decomposition could be sometimes discrete, sometimes continuous. In the present case it will mix both, displaying a discrete and a continuous part:

$$L^2(\Gamma\backslash G) = H_{\mathrm{disc}} \oplus H_{\mathrm{cont}} \tag{9}$$

where both spaces are G-equivariant (i.e. by the action of the right regular representation ρ).

We will be able to decompose further

$$H_{\mathrm{disc}} = H_{\mathrm{cusp}} \oplus H_{\mathrm{res}}. \tag{10}$$

Moreover, H_{cont} will be the analogous of the direct integral decomposition above, and the analogue of $e(\xi t)$, the basis along with we "project" the continuous part, will be the Eisenstein series.

Recall the Iwasawa decomposition $G = NAK$. The principal series representations, for each $s \in \mathbf{C}$, is defined by

$$P_s = \left\{ \eta : G \to \mathbf{C} \ : \ \eta\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} g \right) = y^{1-s}\eta(g) \right\}. \tag{11}$$

This can be formulated in terms of Eisenstein series. We consider on them the "inner product on $L^2(K)$" (since these are totally determined on $B = NA$):

$$\langle \eta_1, \eta_2 \rangle_{P_s} = \int_{NA\backslash G} \eta_1(g)\overline{\eta_2}(g) y^{-2+s+\bar{s}} \mathrm{d}g, \tag{12}$$

where the power of y has been included so that the integrated quantity is indeed NA-invariant. On $s = \frac{1}{2} + i\mathbf{R}$, it is y^{-1}.

The principal series P_s is a representation of G (as induced representation from the character $\cdot^{1-s}$ on A trivially extended on $B = NA$), which is unitary for $s \in \frac{1}{2} + i\mathbf{R}$. The action is by the right regular representation

$$\rho_h(\eta)(g) = \eta(gh). \tag{13}$$

Let $\psi \in C_c^\infty(N\backslash G)$ and introduce

$$\theta_\psi(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\gamma g) \tag{14}$$

the incomplete theta/Eisenstein series, where Γ_∞ is $\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ & 1 \end{smallmatrix}\right)$, the stabilizer of the ∞-cusp. Note that the sum is finite since Γ is discrete and ψ is compactly supported. Typically, for $\psi(g) = y^s$ we will recover the classical Eisenstein series $E_s(g)$. Introduce the space

$$\Theta = \overline{\langle \theta_\psi \ : \ \psi \in C_c^\infty(N\backslash G) \rangle} \tag{15}$$

which is preserved by ρ: we have $\rho_h \theta_\psi = \theta_{\rho_h \theta}$.

The zeroth Fourier coefficient of θ_ψ is defined by

$$\theta_\psi^0(g) = \int_{\Gamma_\infty \backslash N \simeq \mathbf{Z}\backslash \mathbf{R}} \theta_\psi(ng) \mathrm{d}n. \tag{16}$$

Then we have a map (a "projection")

$$\Theta \longrightarrow P_s \tag{17}$$

given by the Mellin transform:

$$\widehat{\theta_\psi}(g, s) = \int_0^\infty \theta_\psi^0(a(y)g) y^{s-1} \frac{\mathrm{d}y}{y}. \tag{18}$$

Note that $\mathrm{d}y/y$ is the Haar measure on the multiplicative group. This converges absolutely for $\Re(s) > 1$, and we can check that

$$\widehat{\theta_\psi}(g, s) \in P_s \tag{19}$$

by a direct change of variables. Selberg continued this theta series up to the critical line $\frac{1}{2} + i\mathbf{R}$, making sense of the "projection" to the unitary principal series. When these Mellin transforms are nontrivial, the original function can be recovered from it.

When $\widehat{\theta_\psi}(g, s) = 0$ we will have a discrete decompositon: these are the *cusp forms.* The remainder will be decomposed through "Mellin inversion", as a direct integral (against Eisenstein series).

In fact, the space of cusp forms is the orthogonal complement of Θ.

Lemma. Let ψ be a smooth function on $N\backslash G$ with compact support and $f \in L^2(\Gamma\backslash G)$ an automorphic function. Then

$$(\theta_\psi, f)_{\Gamma\backslash G} = \frac{1}{2}(\psi, f^0)_{N\backslash G}.$$

In particular, if f is orthogonal to the space Θ, then f is a cusp form.

This is an application of the *unfolding method*, which is valid under our analytic conditions. We have

$$\begin{aligned}
\langle \theta_\psi, f\rangle_{\Gamma\backslash G} &= \int_{\Gamma\backslash G} \overline{f(g)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\gamma g)dg \\
&= \int_{\Gamma_\infty\backslash G} \psi(g)\overline{f(g)}dg \\
&= \int_{N\backslash G}\int_{\Gamma_\infty\backslash N} \psi(ng)\overline{f(ng)}dndg \\
&= \int_{N\backslash G} \psi(g)\overline{f^0(g)}dg = \langle \psi, f^0\rangle_{N\backslash G}.
\end{aligned}$$

If $\langle \theta_\psi, f\rangle = 0$ for all $\psi \in C_c^\infty(N\backslash G)$, then $f^0 = 0$, since we have $\langle \psi, f^0\rangle = 0$ and $C_c^\infty(N\backslash G)$ is a dense subspace.

The discreteness of the space of cusp forms follows from an argument using compact operators. Recall the spectral theorem for Hermitian compact operators, stating that the space on which such an operator acts decomposes as an orthogonal sum of the eigenspaces of the operator. In short, we construct operators that act on the space of cusp forms as compact operators and deduce from there the decomposition into a discrete sum of G-invariant spaces.

Perhaps the space of forms that are right K-invariant provides a picture closer to the intuition built in Fourier analysis. These are functions on the familiar upper half plane $G/K \cong \mathbb{H}$, where the role of the right regular representation is classically taken up by the hyperbolic Laplace operator. This operator commutes with the right regular representation and its spectral theory is analogous to the above decomposition.

Indeed, the space of cusp forms has a basis of eigenfunctions of the Laplacian, say $\{e_i \mid i \in \mathbb{N}\}$. Together with the constant function, these form the discrete spectrum. The continuous spectrum is described by the classical Eisenstein series, defined as

$$E_s(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s,$$

which are eigenfunctions of the Laplace operator, but not square-integrable. A function $f \in L^2(\Gamma\backslash\mathbb{H})$ can be decomposed as

$$f = \sum_{i\in\mathbb{N}} \langle f, e_i\rangle e_i + \langle f, \sqrt{3/\pi}\rangle\sqrt{3/\pi} + \frac{1}{4\pi i}\int_{\Re s=\frac{1}{2}} \langle f, E_s\rangle E_s ds.$$

The space of constant functions is called the residual spectrum because it is generated by residues of the Eisenstein series. An automorphic form orthogonal to the space of cusp forms (so in the space Θ in the previous context) could have been written by Mellin inversion as an integral

$$\int_{\Re s=\sigma} \langle f, E_s\rangle E_s ds,$$

where $\sigma > 1$ a priori. The proper decomposition is then realised by analytically continuing the Eisenstein series and shifting the contour of integration to the line $\Re s = 1/2$, as in the decomposition above. The Eisenstein series have certain poles, which we pick up when shifting the contour, giving rise to the residual spectrum, which one can prove is discrete. In the case of $\mathrm{SL}_2(\mathbb{Z})$, there is only one pole at $s = 1$, giving rise to the constant function.

4. Some generalisations

If instead of $\mathrm{SL}_2(\mathbb{Z})$ we took a different arithmetic subgroups, the decomposition can change in different ways. There are subgroups which are cocompact (e.g. those coming from indefinite quaternion algebras), which only have a discrete spectrum (just like the circle $\mathbb{R}/\mathbb{Z}$, which is compact, has discrete spectrum).

Other groups like the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ still have a continuous spectrum, but it is generated by several Eisenstein series. These subgroups have more than one cusp, and to each cusp we can attach Eisenstein series, $(E_i(s))$. They can be analytically continued and their functional equation now relates the whole vector of Eisenstein series to itself by a matrix (so-called scattering matrix). In the spectral decomposition we need to add integrals over all these different Eisenstein series, which then also contributed to a more complicated residual spectrum.

For going to higher rank, e.g. for $G = \mathrm{SL}_n(\mathbb{R})$, the argument for the discreteness of the space of cusp forms does not change much. Yet the theory of Eisenstein series becomes more complex and was worked out in great generality by Langlands. In short, we need to define Eisenstein series for each class of parabolic subgroups, which are generalisations of the group N above. Moreover, there are ways of embedding $\mathrm{SL}_m(\mathbb{R})$ for $m < n$ into G, which leads to an appearance of the cuspidal spectrum of these lower rank groups in the Eisenstein spectrum.