LECTURE 3 — CLASSIFICATION OF REPRESENTATIONS AND AUTOMORPHIC FORMS

TALK BY FARRELL BRUMLEY

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ABSTRACT. We shall discuss the representation theory of $\mathrm{SL}_2(\mathbf{R})$, linking it with the classical theory of automorphic forms on the upper half-plane.

We will be talking about unitary representations in general before focusing on $\mathrm{SL}(2,\mathbf{R})$.

1. Unitary representations (in general)

Let G be a locally compact topological group.

If G is a compact Lie group, Peter-Weyl theory gives all irreducible representations (they are finite-dimensional and unitary, what can be seen by averaging the inner product over the group). If G is not compact, there are not "enough" finite-dimensional representations. However, infinite-dimensional representations are much more thorny, for instance since we need to be precise about the underlying topological vector spaces.

Definition. A *unitary representation* of G is a homomorphism $\pi : G \to \mathrm{Aut}(H)$ (invertible *bounded* linear operators), where H is a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ preserved under π (in other words, $\pi : G \to U(H)$), and so that $(g, v) \mapsto \pi(g)v$ is continuous under the strong topology (coming from the norm) on H.

Remarks. This definition motivates some discussion:

- Such π are sometimes called strongly continuous.
- The continuity of the action map is equivalent to"separate continuity", in each variable: for each g, the map $v \mapsto \pi(g)v$ is continuous (this is automatic since $\pi(g)$ is bounded linear) and for each v, the map $g \mapsto \pi(g)v$ is continuous.
- We are *not* asking that $\pi : G \to \mathrm{Aut}(H)$ be continuous, where $\mathrm{Aut}(H)$ is given the topology induced by the operator norm. Doing so would eliminate (if G is not discrete) the right regular representation $g \mapsto R_g$ of G on $L^2(G)$ (i.e. right translations), which is central in harmonic analysis.

  Proof. Here is why. Let $U \ni 1$ be a compact subset and $g \in U \backslash \{1\}$. Let $f \in C_c(G)$ nonnegative and $\|f\|_2 = 1$ such that $\mathrm{supp}(f) \cap \mathrm{supp}(f)g = \varnothing$. Since R_g is unitary (with a right invariant Haar measure), we immediately deduce $\|R_g f\|_2 = 1$. Then

$$\begin{aligned}\|R_g f - f\|_2^2 &= \langle R_g f - f, R_g f, f\rangle \\ &= \|R_g f\|_2^2 - 2\langle R_g f, f\rangle + \|f\|_2^2 = 2\end{aligned}$$

Date: October 14, 2021.

Notes updated on October 19, 2021

These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

since the supports are disjoint so that the central inner product vanishes. So that necessarily, we have the lower bound $\|R_g - 1\|_{\mathrm{op}} \geqslant \sqrt{2} > 0$ for all $g \neq 1$. □

Definition. A representation π is said to be *irreducible* if there is no closed (proper, nonzero) invariant subspaces.

Remark. Note that any unitary representation is totally reducible (i.e. semisimple): indeed, taking an invariant subspace V, its Hilbert orthogonal complement $V^\perp$ would be also closed and invariant.

Definition. An *isometric interwiner* between two unitary representations $\pi, \pi' : G \to U(H)$ is an isometry $\phi \in U(H)$ such that $\phi \circ \pi = \pi'$. We say that π and π' are *unitarily equivalent.*

Let $\widehat{G}$ denote the set of equivalence classes of unitarily equivalent irreducible unitary representations of G, called the *unitary dual of* G. It is particularly difficult to study, even for real semisimple Lie groups.

The topology on the unitary dual is the *Fell topology*: π and π' are close if their matrix coefficients defined by $\xi^\pi_{v,w} : g \mapsto \langle \pi(g)v, w\rangle$ are close on compacta. The Fell topology is typically non-Hausdorff (even in the case of $\mathrm{SL}(2, \mathbf{R})$).

Remark. For a slightly smaller class of groups (viz. *type I*), one can define a (Plancherel) measure μ^{Pl} on $\widehat{G}$, allowing to do harmonic analysis, i.e. so that we have for all $f \in C_c(G)$ the inversion formula

$$f(1) = \int_{\widehat{G}} \widehat{f}(\pi) \mathrm{d}\mu^{\mathrm{Pl}}\pi. \tag{1}$$

A typical question is: what is the support of the Plancherel measure?

Remark. In fact, even if we are only interested in unitary representations, to understand them one must look beyond them. We shall broaden the setting of study to have a nicer, better behaved, class of representations. One motivation is the following. We have the *subrepresentation theorem* of Casselman: for G real reductive Lie group, any irreducible unitary representation can be embedded as a subrepresentation of a (non-unitary) principal series representation (see the $\mathrm{SL}(2, \mathbf{R})$ setting). Thus, non-unitary representations (and their subobjects) are still interesting to study.

2. The $\mathrm{SL}(2, \mathbf{R})$ case

Lemma. If π is a finite-dimensional unitary representation of $\mathrm{SL}(2, \mathbf{R})$, then $\pi = 1$.

Proof. Say $\pi : G \to U(n)$. By the *LU* decomposition let's look only at upper-triangular matrices. Sinca all $\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$ are conjugate in $\mathrm{SL}(2, \mathbf{R})$ for $x > 0$, one of the $\pi\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$ is in $U(n)$. But conjugacy classes in $U(n)$ are closed, so that

$$\lim_{x \to 0} \pi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \pi \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 1 \tag{2}$$

is in the same conjugacy class as $\pi\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$. Thus the conjugacy class is $\{1\}$. Hence $\pi = 1$ on upper triangular matrices. The same happens symmetrically on lower triangular matrices. And use the fact that $\mathrm{SL}(2, \mathbf{R})$ is generated by both. □

Fact. The finite-dimensional irreducible non-unitary representations of $G = \mathrm{SL}(2, \mathbf{R})$ are the symmetric powers $\mathrm{Sym}^n : G \to \mathrm{Aut}(P_n)$, where P_n is the space of homogeneous polynomials in two variables z_1, z_2 of degree n, and the action is given by

$$\mathrm{Sym}^n\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix} P\right)\begin{pmatrix} z_1 \\ z_2\end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} z_1 \\ z_2\end{pmatrix}\right). \tag{3}$$

2.1. **Principal series representations.** Let $G = \mathrm{SL}_2(\mathbf{R})$, $K = \mathrm{SO}(2)$ and $B = \left(\begin{smallmatrix} \star & \star \\ & \star\end{smallmatrix}\right)$. The Iwasawa decomposition writes $G = BK$. Let's induce representations from the "big" (in the sense that the quotient is compact) subgroup B of G. Let $s \in \mathbf{C}$ and $\varepsilon \in \{0,1\}$: these will be the parameters of the continuous series of representations.

Introduce $\delta\left(\begin{smallmatrix} a & b \\ & a^{-1}\end{smallmatrix}\right) = |a|^2$ (the modular quasicharacter, taking into account the fact that G is not unimodular). Let

$$U_{s,\varepsilon} = \left\{ f \in C^\infty(G) \ : \ f\left(\left(\begin{smallmatrix} a & b \\ & a^{-1}\end{smallmatrix}\right) g\right) = \mathrm{sgn}(a)^\varepsilon \delta(a)^{1/2}|a|^s f(g)\right\}. \tag{4}$$

Define a norm (and thus an inner product) on f by

$$\|f\|_2^2 = \int_K |f(k)|^2 \mathrm{d}k. \tag{5}$$

It is not unnatural, since f has a prescribed behavior under the action of B, so the leeway only exists on K. Let $H_{s,\varepsilon}$ be the Hilbert space completion of $U_{s,\varepsilon}$. G acts on $H_{s,\varepsilon}$ by right translations.

Lemma. We have the following:

- For $s \in i\mathbf{R}$, the space $H_{s,\varepsilon}$ is a unitary representation (for the above inner product). They are irreducible except $H_{0,1}$. They are isomorphic under $s \mapsto -s$, inequivalent otherwise.
- For $\varepsilon = 0$, the space $H_{s,0}$ is unitarizable for $s \in (-1,1)\backslash\{0\}$ for some other (strange) inner product. They are irreducible. They are isomorphic under $s \mapsto -s$, inequivalent otherwise.
- Otherwise, $H_{s,\varepsilon}$ are not unitarizable.

Proof. For the irreducibility, it is enough to show that the restriction to the lower triangular subgroup (the opposite Borel) $\bar{B}$ is irreducible. For that we invoke Schur's lemma for unitary representations. We thus let L be a bounded linear operator commuting with the $\bar{B}$-action: we must show that L is a scalar. Functions f in the principal series representation are functions on $N\backslash G = \mathbf{R}^2 - \{0\}$, which are $(1+s)$-homogeneous under the A-action. We may restrict them to the line $(\mathbf{R}, 1)$; the resulting space is called the *line model*: $\varphi(x) := f(x,1)$. We have $\left(\begin{smallmatrix} 1 & \\ y & 1\end{smallmatrix}\right)\varphi(x) = \varphi(x-y)$ and $\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right)\varphi(x) = |a|^{1+s}\varphi(a^2 x)$. Since L commutes with the first of these actions, $\hat{L}$ acts on the Fourier transform $\hat{\varphi}$ by multiplication by a bounded measurable function m. To show that m is constant, we use the fact that L commutes with the second action to show that $m(a^{-2}x) = m(x)$ for all a and almost every x. □

[Picture to be drawn someday, for each ε]

This is not the end of the story, and we still have a whole series of representations: the *discrete series.* They can be found as subrepresentations (recall Casselman's subrepresentation theorem) of non-unitary principal series $H_{k,\varepsilon}$ for $k \geqslant 2$, and $\varepsilon \equiv k \bmod 2$. More precisely, for integers $k \geqslant 2$,

$$0 \longrightarrow D_k^+ \oplus D_k^- \longrightarrow H_{k-1,\varepsilon} \longrightarrow \mathrm{Sym}^{k-2} \longrightarrow 0 \tag{6}$$

and the subrepresentation splits into a holomorphic $(+)$ discrete series and an antiholomorphic discrete series $(-)$. They have an explicit description. There is also the trivial representation. This describes the whole unitary dual of $\mathrm{SL}(2,\mathbf{R})$.

Remarks. Some remarks:

- The support of the Plancherel measure is made of the tempered unitary representations, i.e. everything but the trivial representation and the complementary series $H_{s,0}$ for $s \in (0,1)$.
- Link to classical modular forms:
  - weight zero Maass forms: $H_{s,0}$ for $s \in i\mathbf{R} \cup (-1,1)$
  - weight one Maass forms: $H_{s,1}$ for $s \in i\mathbf{R}\backslash\{0\}$
  - weight one holomorphic modular forms: $D_1 = D_1^+ \oplus D_1^-$, limits of discrete series, irreducible on $\mathrm{GL}(2,\mathbf{R})$
  - weight $k \geqslant 2$ holomorphic modular forms: $D_k = D_k^+ \oplus D_k^-$, irreducible on $\mathrm{GL}(2,\mathbf{R})$