LECTURE 2 — INTEGRAL REPRESENTATIONS OF L-FUNCTIONS

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Abstract. We will start by talking about the Riemann zeta function and how it can almost be written as a Mellin integral of a nice function called theta series. Then we will try to give a big picture on what one can do for higher rank L-functions. In the process we will have ideas on automorphic forms and their Fourier expansions.

The spirit of integral representations of L-functions stems from the most basic example: the Riemann zeta function.

1. Riemann zeta function

Define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1 \tag{1}$$

We have a meromorphic continuation of $\zeta(s)$ to the whole complex place $\mathbf{C}$. An usual way of doing it in standard books is the following. For $x \geqslant 0$, we consider the partial sum

$$\sum_{n \leqslant x} \frac{1}{n^s} \tag{2}$$

There are formulas allowing to estimate such partial sums, namely the Euler-Maclaurin formula:

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} \mathrm{d}t \tag{3}$$

The left hand side defines the $\zeta(s)$ when x goes to ∞. The right hand side has a meaning for $\Re(s) > 0$ since the integral is absolutely convergent. This provides the meromorphic continuation of $\zeta(s)$ to the region $\Re(s) > 0$.

1.1. Functional equation.

Define $\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ where Γ is the Euler gamma function. Introduce the completed zeta function

$$\xi(s) = \Gamma_{\mathbf{R}}(s)\zeta(s). \tag{4}$$

The functional equation states

$$\xi(s) = \xi(1-s). \tag{5}$$

The functional equation allows to continue $\zeta(s)$ meromorphically to the whole complex plane. The meromorphic continuation and the functional equation are two deep facts about $\zeta(s)$. If we

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These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

consider generalizations of $\zeta(s)$, e.g. L-functions, do we have also the similar properties? with similar proofs?

For Dirichlet series, it is not the case in general: the coefficients should be very specific, typically coming from automorphic forms (it is even almost equivalent, by Weil's converse theorems). We would like to understand where the properties of automorphic forms come into play to get these properties.

2. Jacobi θ series

Define the Poincaré upper half-plane $\mathcal{H} = \{z \in \mathbf{C} \; : \; \Im(z) > 0\}$. Introduce

$$\theta(z) = \sum_{n \in \mathbf{Z}} e^{i\pi n^2 z}, \quad z \in \mathcal{H}. \tag{6}$$

It is bounded by triangular inequality:

$$\theta(z) \leqslant \sum_{n \in \mathbf{Z}} e^{-\pi n^2 \Im(z)}, \quad z \in \mathcal{H}, \tag{7}$$

and does converge since $\Im(z) > 0$. It hence defines a holomorphic function on $\mathcal{H}$.

Note that there is a symmetry in $n \leftrightarrow -n$, so this motivates to split

$$\theta(z) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi n^2 z}. \tag{8}$$

Note that the sum appearing on the right has exponential decay when n grows, which endows it with much better analytic behavior than the whole $\theta(z)$. This is a way to *regularize* $\theta(z)$, removing the singularity (non-integrable part) arising in the zeroth Fourier coefficient.

Consider the (regularized) integral

$$\int_0^{\infty} \frac{\theta(iy) - 1}{2} y^{s/2} \frac{\mathrm{d}y}{y}, \tag{9}$$

which is just the Mellin transform of the sum appearing above in the splitting of $\theta(z)$. Note that the measure $\mathrm{d}y/y$ is a *multiplicative* Haar measure. The integral is explicitly

$$\int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{\mathrm{d}y}{y} \tag{10}$$

We want to switch the summation and the integration. To do so, let's cut the integral at 1 to treat separately the behaviors at 0 and and ∞. For $y \to \infty$, the sum is dominated by a geometric series (hence its first term), hence the integral is dominated by

$$\int_1^{\infty} e^{-y} y^{s/2} \frac{\mathrm{d}y}{y} \tag{11}$$

which converges as long as $\Re(s) > 0$.

Let us show that it is also convergent when $y \to 0$. The sum there is not trivial to dominate, indeed bounding $e^{-\pi n^2 y}$ by $e^{-\pi n y}$ would give a geometric series and the sum would be bounded by $1/(1 - e^{-\pi y}) \sim 1/y$. We could only conclude that the integral is absolutely convergent for $\Re(s) > 2$ which is even worse than the original domain. We can do better by using the Poisson summation formula (and the theta identity, see below), and bound the sum by $1/\sqrt{y}$, so that the integral is dominated by

$$\int_0^1 y^{(s-1)/2} \frac{\mathrm{d}y}{y} \tag{12}$$

and $y^{s/2-3}$ is integrable near 0 for $\Re(s) > 1$. In this domain, we can switch summation, change variable and get

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{\mathrm{d}y}{y} &= \sum_{n=1}^{\infty} \frac{1}{(\pi n^2)^{s/2}} \int_0^{\infty} e^{-z} z^{s/2} \frac{\mathrm{d}z}{z} \\ &= \pi^{-s/2} \zeta(s) \Gamma(s/2). \end{aligned}$$

Finally, we end up an *integral representation* of the ξ function

$$\int_0^{\infty} \frac{\theta(iy) - 1}{2} y^{s/2} \frac{\mathrm{d}y}{y} = \Gamma_{\mathbf{R}}(s) \zeta(s), \quad \Re(s) > 1. \tag{13}$$

3. Modularity

The theta series is endowed with strong symmetry properties.

3.1. **Poisson summation formula.**

For a Schwartz function f on $\mathbf{R}$, we have the Poisson summation formula (note we can take stronger assumption, e.g. compactly supported, or weaker, e.g. three first derivatives integrable, cf. Finis-Lapid)

$$\sum_{n \in \mathbf{Z}} f(z) = \sum_{n \in \mathbf{Z}} \hat{f}(n) \tag{14}$$

where $\hat{f}$ is the Fourier transform of f, i.e.

$$\hat{f}(t) = \int_{\mathbf{R}} f(x) e^{-2i\pi x t} \mathrm{d}x. \tag{15}$$

Note that there is sometimes leverage on the normalizations taken for the Fourier transform. Here we want the Poisson summation formula to hold as stated, and the Fourier transform and Poisson summation to be involutive operators, and this determines the normalization. The proof is typical, and comes from the Fourier expansion

$$g(x) := \sum_{n \in \mathbf{Z}} f(x + n) = \sum_{n \in \mathbf{Z}} \hat{g}(n) e^{2i\pi n x} \tag{16}$$

and taking $x = 0$.

From this formula we deduce a functional equation of the Jacobi θ series. Indeed, taking the Gaussian $f_z(x) = e^{i\pi x^2 z}$ for $z \in \mathcal{H}$. Its Fourier transform is

$$\hat{f}(x) = \frac{1}{\sqrt{-iz}} e^{-i\pi \xi^2/z}. \tag{17}$$

Then we use the Poisson summation formula and get, summing over $n \in \mathbf{Z}$,

$$\theta(z) = \frac{1}{\sqrt{-iz}} \theta(-1/z). \tag{18}$$

This is the *automorphy/modularity property* of θ, and we saw it is nothing more than the Poisson summation formula. Coming back to the integral representation above, we cut the integral at 1:

$$\Gamma_{\mathbf{R}}(s)\zeta(s) = \int_0^\infty \frac{\theta(iy)-1}{2} y^{s/2} \frac{\mathrm{d}y}{y} = \int_0^1 + \int_1^\infty \tag{19}$$

The second integral has no analytic issue since the regularized θ has exponential decay at ∞, and thus defines an entire function (i.e. holomorphic for all $s \in \mathbf{C}$).

For the first one, change variable $y \mapsto 1/y$ and use the modularity property:

$$\begin{aligned}\int_0^1 \frac{\theta(iy)-1}{2} y^{s/2} \frac{\mathrm{d}y}{y} &= \int_1^\infty \frac{\theta(i/y)-1}{2} y^{-s/2} \frac{\mathrm{d}y}{y} \\ &= \int_1^\infty \frac{\theta(iy)\sqrt{y}-1}{2} y^{-s/2} \frac{\mathrm{d}y}{y}.\end{aligned}$$

Use the decomposition $\theta = \theta - 1 + 1$. The first part decays rapidly thanks to the exponential decay of $\theta - 1$ and gives an entire integral:

$$\int_1^\infty \frac{\theta(iy)-1}{2} y^{\frac{1-s}{2}} \frac{\mathrm{d}y}{y}. \tag{20}$$

The second part is

$$\int_1^\infty \frac{\sqrt{y}-1}{2} y^{-s/2} \frac{\mathrm{d}y}{y} = \frac{1}{s-1} + \frac{1}{s} \tag{21}$$

which is meromorphic for $\Re(s) > 0$. Finally, the integral representation of $\xi(s)$ can be written as an entire part plus $1/s + 1/(s-1)$, so we see the meromorphic function with its simple poles and residues explicitly:

$$\Gamma_{\mathbf{R}}(s)\zeta(s) = \frac{1}{s-1} + \frac{1}{s} + \int_0^\infty \frac{\theta(iy)-1}{2} y^{s/2} \frac{\mathrm{d}y}{y} + \int_1^\infty \frac{\theta(iy)-1}{2} y^{\frac{1-s}{2}} \frac{\mathrm{d}y}{y}. \tag{22}$$

We also conclude to the functional equation, since the expression is symmetric in $s \mapsto 1-s$.

4. Openings towards general integral representations

We finally proved the meromorphic continuation and functional equation of ζ. Instead of doing partial summations, we translated the properties of θ into properties of its Mellin transform (which is ξ, the completed zeta function). This is the general philosophy of integral representations, that we illustrated here for GL(1) L-functions. We integrated two automorphic forms: $\frac{1}{2}(\theta(iy)-1)$ and $y^{s/2}$, getting a kind of Rankin-Selberg GL(1) $\times$ GL(2) to represents a degree one L-function. (Yet we should not think of θ as a GL(2) automorphic form since it gives a GL(1) L-function in the end.)

For GL(n), we also have analogous integral representations that generalize this Mellin integral. They are based on Whittaker functions coming from Fourier expansions, which are in turn integrated against GL(1) objects.

Question. More general group beyond GL(n)? Different integral representations?

There are Whittaker models in generic cases (by definition), but beyond GL and SO (Soudry, Ginzburg) there is no know relation between them and the attached automorphic L-function.

Question. What is the analogous of the "modularity" condition?

In fact it is the equation/automorphicity with respect to the long Weyl element that translates into the functional equation, through Mellin transform. Note it is the same as in the above GL(1) case, where also the functional equation comes from the modularity property of θ, which is a transformation law under $z \mapsto -1/z$ (the long Weyl element action on $\mathcal{H}$).

Introduce the non-holomorphic Eisenstein series $E(z,s)$ for $z \in \mathcal{H}$ and $s \in \mathbf{C}$, in the $\mathrm{GL}(2,\mathbf{R})$ case. The main formula will be of the form

$$\int_0^\infty \left[E_0(iy,s) - E_0(iy,0)\right] y^s \frac{\mathrm{d}y}{y} = \xi(s)\xi(-s) \tag{23}$$

where E_0 is the zeroth Fourier coefficient.