LECTURE 1 — AUTOMORPHIC REPRESENTATIONS

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Abstract. In this introductory talk, assuming some familiarity with modular forms, we will present the representation theoretic framework where automorphic forms live.

The central topic of this reading group will be automorphic forms, and in particular their representation theoretic aspects. We will start from pretty abstract definitions and backtrack to show their consistency with classical objects.

1. Automorphic forms

Let G a reductive group over $\mathbf{Q}$ (we will mostly put emphasis on $\mathrm{GL}_n/F = \mathrm{Res}_{\mathbf{Q}}^F(\mathrm{GL}_n)$ where F is a number field). Let K_v° be the standard maximal open compact subgroup. For finite places v, it is $G(\mathfrak{O}_v)$. For archimedean places $v \mid \infty$, it is the connected component of the identity in a maximal compact subgroup. In the case of $G = \mathrm{GL}_n$, it is $K_v^\circ = \mathrm{SO}_n(\mathbf{R})$ over $\mathbf{R}$ and $K_v^\circ = \mathrm{U}_n$ over $\mathbf{C}$. In GL_n, the maximal open compact subgroups are all conjugate, but not for other groups like SL_n, GSp_4, etc. Let $K^\circ = \prod_v K_v^\circ$.

Let $\mathbf{A}$ be the adeles of F. Let $G(\mathbf{A}) = \prod_v' G(F_v) = \varinjlim \prod_{v\in S} G(F_v) \prod_{v\notin S} K_v^\circ$ where the inductive limit is taken over finite set of places S.

Definition. A function $\phi : G(\mathbf{A}) \to \mathbf{C}$ is said to be an *automorphic form* if, writing $g = g_f g_\infty$,

- (smoothness) ϕ is smooth (C^∞) at archimedean places, i.e. as function of G_∞
- (smoothness) ϕ is smooth (locally constant) at finite places, i.e. as function of G_f
- (automorphy) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(\mathbf{Q})$
- ϕ is K-finite (weight condition), i.e. $\langle \phi(\cdot k) \ : \ k \in K^\circ \rangle$ is finite dimensional
- (nearly holomorphic) Letting $\mathcal{Z}$ the center of the envelopping Lie algebra $U(\mathrm{Lie}(G(F_\infty)))$, the space $\langle X \cdot \phi : X \in \mathcal{Z} \rangle$ is finite dimensional
- (growth) For any norm $\|\cdot\|$, there is an r such that $\phi(g) \ll \|g\|^r$ (QUESTION : norm as an algebraic group ? do we consider the inverse of the determinant ? Probably, see Getz Hahn. Do we see it on classical modular forms? WHy? And what meaning of the norm for a general reductive group? Is the embedding)

Question. what about the center ? Do we have an $F_\infty^\times$ action that spans an finite-dimensional vector space? do we need to assume something or is it a consequence that, under the center, it is also finite-dimensional? Or can we have any continuous character?

Remark. An *irreducible* admissible representation of $G(\mathbf{A})$ has necessarily a central character. In practice we will work with a fixed central character ω.

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These notes have been taken on the go by Didier Lesesvre. In particular they may contain typos and errors.

2. Spaces of automorphic forms

The space of automorphic forms $\mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A}),\omega)$ can be split into

$$\mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A}),\omega) = \bigoplus \mathcal{A}(\tau, K, \mathcal{I}, \omega). \tag{1}$$

Fix the following data

- (K_∞°-type) τ a (finite dimensional) ireducible representation of K_∞°. $V_\tau \simeq \mathbf{C}^{\dim(\tau)}$
- (central character) $\omega : F^\times\backslash\mathbf{A}^\times$ a Hecke character
- (holomorphy condition) $\mathcal{I} \subset \mathcal{Z}$ a finite codimensional ideal
- (level) $K_f \subset K_f^\circ$ an open compact subgroup of finite index in K_f°

The space $\mathcal{A}(\tau, K, \mathcal{I}, \omega) \subset \mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A}))$ consists of $\phi : G(\mathbf{Q})\backslash G(\mathbf{A}) \to V_\tau$ such that

- (weight condition) for all $k \in K_\infty^\circ$, $\phi(gk) = \tau(k)\phi(g)$
- (right-K_f-invariance) for all $k_f \in K_f$, $\phi(gk_f) = \phi(g)$, i.e. ϕ factors though $G(\mathbf{Q})\backslash G(\mathbf{A})/K$
- (central character) for all $z \in Z \simeq \mathbf{A}^\times \subset G(\mathbf{A})$ (diagonally embedded), $\phi(zg) = \omega(z)\phi(g)$
- (differential equations) $\mathcal{I} \cdot \phi = 0$

Fix a unitary central character $\omega : F^\times\backslash\mathbf{A}^\times \to S^1$. We define

$$L^2(G(\mathbf{Q})\backslash G(\mathbf{A}),\omega) = \left\{ \phi \in \mathcal{A}([G],\omega) \ : \ \int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} |\phi(g)|^2 dg < \infty \right\}. \tag{2}$$

Note that fixing the central character is necessary, otherwise it is never possible: the center is not compact, so the integral would diverge. To quotient by the center (i.e. to have something well-defined above), we need a central character.

3. Cuspforms

Definition. $\phi \in \mathcal{A}$ is a *cuspform* if for any parabolic subgroup (enough to do it for a maximal parabolic) which Levi decomposition is $P = MU$, and for all $g \in G(\mathbf{A})$,

$$\int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \phi(ug)du = 0. \tag{3}$$

A theorem of Gelfand and Piatetskii-Shapiro ensures that the growth condition from above is automatically satisfied.

4. Relation with classical modular forms

The relation of these adelic automorphic forms with classical modular forms derive from the following facts:

- $G(\mathbf{Q})$ is a lattice in $G(\mathbf{A})$, the same way that $\mathbf{Z}$ is a lattice in $\mathbf{R}$. So the above cusp condition can be rephrased as a Fourier coefficient condition.
- For $g \in G(\mathbf{A})$, $G(\mathbf{Q}) \cap gK_fG(F_\infty)g^{-1} =: \Gamma$ is a lattice, i.e. a discrete subgroup of $G(F_\infty)$ of finite covolume (called a *congruence subgroup*).
- Strong approximation theorem: for any simple simply connected group G^1 (such as SL_n), for any place v we have that $G^1(\mathbf{Q})G^1(F_v)$ is dense in $G^1(\mathbf{A})$.

5. Relation with locally symmetric spaces

The theory can also be linked to algebraic topology and Shimura varieties. The space of automorphic forms is endowed with an action of the Hecke algebra, which factors through the algebra of bi-K_v°-invariant functions of the Hecke algebra. This is a polynomial algebra (e.g. for GL_2 it is the polynomials in the classical Hecke operators T_v).

Consider $S_K = G(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty^\circ Z_\infty^\circ$ is a locally symmetric space (real orbifold, a manifold if K_f is small). You can then consider the singular/Betti cohomology $H^\bullet(S_{K_f}, \mathbf{C})$, which has the same action by the Hecke operator. Taking the limit over all K_f's, it is acted on by $G(\mathbf{A}_f)$ and the $(\mathrm{Lie}(G(F_\infty)), K)$-action.

A theorem by Borel-Wallach decomposes this as a sum of automorphic representations. (Bridge towards other topics, seeing automorphic forms as cocyles).