SUMS OF ASCENDING EVEN POWERS

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Freiman and Scourfield proved that any large enough integer can be written as a sum of a certain number of ascending even powers. We use the circle method to provide the first explicit bound on this number, and show that any large enough integer can be written as a sum of 173 ascending even powers.

1. INTRODUCTION

1.1 ASCENDING POWERS

Since the first steps of the circle method at the beginning of the twentieth century, a wide literature has been published concerning the Waring problem and its generalizations of mixed types. The question is the one of representing a large enough integer n in the form

$$n = x_1^{k_1} + \cdots + x_s^{k_s}, \qquad k_1 \leqslant \cdots \leqslant k_s, \qquad x_i, k_i \in \mathbf{N}. \tag{1.1}$$

Freiman [Fre49] and Scourfield [Sco60] characterized the sequences of powers for which (1.1) is solvable for all large enough integer n. More precisely, they established the following result.

Theorem 1 (Freiman-Scourfield). *Let $(k_i)_i$ be a non-decreasing sequence of positive integers. The series $\sum k_i^{-1}$ is divergent if and only if for any $j \in \mathbf{N}$, there exists an $s \geqslant j$ such that every large enough integer n is representable as*

$$n = x_j^{k_j} + x_{j+1}^{k_{j+1}} + \cdots + x_s^{k_s}, \qquad x_j, \ldots, x_s \in \mathbf{N}. \tag{1.2}$$

Despite this result, few is known concerning the lowest possible s for which (1.2) holds provided the series $\sum k_i^{-1}$ is divergent. Many authors, among which Roth [Rot49, Rot51], Thanigasalam [Tha68, Tha80], Vaughan [Vau70, Vau71], Brüdern [Brü87, Brü88] and Ford [For95, For96], struggled for half a century to determine the least s for which any large enough integer can be written as a sum of ascending powers,

$$n = x_2^2 + x_3^3 + \cdots + x_s^s, \qquad x_2, \ldots, x_s \in \mathbf{N}. \tag{1.3}$$

The purpose of this paper is to address another instance of the Waring problem of mixed type, restricted to the sequence of ascending even powers $\{2n\}_{n \in \mathbf{N}}$. This elegant variation has been recently explored by Brüdern. The set of integers representable as

$$n = x_2^2 + x_4^4 + x_6^6, \qquad x_2, x_4, x_6 \in \mathbf{N}, \tag{1.4}$$

is of zero density since the sum of reciprocal exponents fails to reach one. The main result of Brüdern's paper [Brü19] implies in particular that the set of integers representable as

$$n = x_2^2 + x_4^4 + x_6^6 + x_8^8, \qquad x_2, x_4, x_6, x_8 \in \mathbf{N}, \tag{1.5}$$

has positive density, but less than 1. A corollary of his result ensures that the set of integers representable as

$$n = x_2^2 + x_4^4 + x_6^6 + x_8^8 + x_{10}^{10}, \qquad x_2, x_4, x_6, x_8, x_{10} \in \mathbf{N}, \tag{1.6}$$

is of density one. This settles the problem of the ascending even powers in the density aspect. The purpose of this paper is to give a bound on the number of even ascending powers necessary to write all but finitely many natural numbers in this form. The main result is the following.

Theorem 2. *Let* $s = 173$. *Every sufficiently large natural number* n *is representable in the form*

$$n = x_2^2 + x_4^4 + x_6^6 + \cdots + x_{2s}^{2s}, \qquad x_2, \dots, x_{2s} \in \mathbf{N}. \tag{1.7}$$

Let $R_s(n)$ be the number of ways of writing n as in (1.7). The aim is to prove that $R_s(n) > 0$ for n large enough, and to find the best possible s for which it happens.

Remark. The theoretical limit of the circle method is given by a sum of reciprocal exponents equal to 2. The quality of the result can henceforth be judged by how close to 2 is the sums of reciprocal exponents. In the case of the growing powers, Ford [For96] reaches the value

$$\sum_{k=2}^{15} \frac{1}{k} \approx 2.32, \tag{1.8}$$

while the result presented here, in the case of the ascending even powers, yields

$$\mu = \sum_{k=1}^{173} \frac{1}{2k} \approx 2.85. \tag{1.9}$$

This is slightly worse than Ford's case, that is no surprise: our sequence is growing faster than the one in (1.8) so that the Weyl inequality, particularly powerful for small powers, cannot be so strongly used in the present case.

We prove in fact a quantitative version of Theorem 2.

Theorem 3. *Let* $s = 173$. *We have, as* n *grows to infinity,*

$$R_s(n) \gg F(0) n^{-1} \asymp n^{\mu - 1}. \tag{1.10}$$

1.2 OUTLOOK OF THE PROOF

This is a typical problem of additive number theory. Similar to the Waring problem or to the full ascending power problem (1.3), the question is amenable to the Hardy-Littlewood circle method. We define the Farey dissection and the major arcs in Section 2, estimate the contribution of minor arcs in Section 3, approximate the generating functions on major arcs in Section 4, prune the major arcs to logarithmically-scaled arcs in Section 5 and finally conclude by estimating the main term in Section 6.

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2. CIRCLE METHOD

2.1 DEFINITION OF MAJOR ARCS

Let $n \geqslant 1$ and $0 < \tau < 1/2$. Set $X = n^\tau$. Denote $\|x\|$ the integer norm of x, that is the distance to the closest integer. For $1 \leqslant a \leqslant q \leqslant X$ with $(a,q) = 1$, introduce the major arc around a/q defined by

$$\mathfrak{M}(X;q,a) = \left\{ \alpha \in [0,1] \; : \left\| \alpha - \frac{a}{q} \right\| \leqslant \frac{X}{qn} \right\}. \tag{2.1}$$

The major arcs $\mathfrak{M}(X)$ consist in the union of all these arcs for $q \leqslant X$, namely

$$\mathfrak{M}(X) = \bigcup_{q \leqslant X} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(X;q,a). \tag{2.2}$$

The major arcs $\mathfrak{M}(X;q,a)$ are pairwise disjoint provided $X < \frac{1}{2}\sqrt{n}$. This is the case for large enough n since $\tau < 1/2$, so that any $\alpha \in \mathfrak{M}(X)$ uniquely determines the associated a and q. We use this fact every time these quantities appear without notice and α is fixed in a major arc. Define the minor arcs $\mathfrak{m}(X)$ as the remaining points on the circle, that is to say

$$\mathfrak{m}(X) = [0,1] \backslash \mathfrak{M}(X). \tag{2.3}$$

2.2 ANALYTIC GENERATING FUNCTIONS

Introduce the dyadic exponential sums, for all $k \geqslant 1$ and $\alpha \in \mathbf{R}$,

$$f_k(\alpha) = \sum_{m \in X_k} e\big(\alpha m^k\big), \qquad \text{where} \quad X_k = \big]\!\big] n^{1/k}, 2n^{1/k} \big]\!\big], \tag{2.4}$$

and their smooth analogues, for a certain $\gamma > 0$ to be determined later,

$$g_k(\alpha) = \sum_{m \in Y_k} e\big(\alpha m^k\big), \qquad \text{where} \quad Y_k = \mathscr{A}\big(n^{1/k}, n^\gamma\big). \tag{2.5}$$

Here, $\mathscr{A}(n^{1/k}, n^{\gamma})$ stands for the n^{γ}-smooth numbers less than $n^{1/k}$, that is to say

$$\mathscr{A}\left(n^{1/k}, n^{\gamma}\right) = \left\{x \leqslant n^{1/k} \ : \ p|x \Longrightarrow p \leqslant n^{\gamma}\right\}. \tag{2.6}$$

Let $K = \{2, 4, \ldots, 2s\}$ be the set of indices we consider in (1.7). For a partition $K = K_f \sqcup K_g$, introduce the generating function

$$F = \prod_{k \in K_f} f_k \prod_{k \in K_g} g_k. \tag{2.7}$$

In this article, we let $K_f = \{2, 4, 6, 8\}$ and $K_g = \{10, \ldots, 2s\}$. In particular, the associated Fourier coefficients

$$r_s(n) = \int_{\mathbf{R}/\mathbf{Z}} F(\alpha) e(-n\alpha) \mathrm{d}\alpha, \qquad n \in \mathbf{N}, \tag{2.8}$$

are less than $R_s(n)$, the number of solutions to the initial problem (1.7). It is therefore sufficient to prove that $r_s(n) > 0$ for n large enough in order to prove Theorem 2. The expected size of $r_s(n)$ is of order $F(0)n^{-1}$, as proven in the last section and stated in Theorem 3. This provides a guide to estimating error terms.

Remark. We would ultimately like to take $K_f = \emptyset$ since mean value theorems for smooth functions provide stronger bounds. However non-smooth functions f_k are necessary in order to apply iterative methods and to be approximated efficiently around rationals.

3. CONTRIBUTION OF MINOR ARCS

3.1 STRATEGY

We begin by bounding the contribution of minor arcs to the integral (2.8). Introduce a partition

$$K = \{2, 8\} \sqcup K_1 \sqcup K_2.$$

Remark. The sets K_1 and K_2 are chosen in order to get the best values in the application of the mean value theorems. See Sections 3.5 and 4.3 for more details.

Introduce

$$F_1 = \prod_{k \in K_1 \cap K_f} f_k \prod_{k \in K_1 \cap K_g} g_k,$$
$$F_2 = \prod_{k \in K_2 \cap K_f} f_k \prod_{k \in K_2 \cap K_g} g_k,$$

so that $F = f_2 F_1 F_2$. By Cauchy's inequality, we have

$$\int_{\mathfrak{m}} F \ll \sup_{\mathfrak{m}} |f_2| \sup_{\mathfrak{m}} |f_8| \left(\int_0^1 |F_1|^2\right)^{1/2} \left(\int_0^1 |F_2|^2\right)^{1/2}. \tag{3.1}$$

The remainder of this section is dedicated to introducing tools in order to bound the terms on the right hand side of (3.1).

3.2 WEYL'S INEQUALITY

To estimate the supremum in the bound (3.1), we need pointwise bounds [Vau97] for f_2 on the minor arcs. These are provided by the following classical lemma.

Lemma 1 (Weyl's inequality)**.** *Let $\alpha \in \mathbf{R}$. Suppose $(a,q) = 1$ and $\alpha \in \mathbf{R}$ such that $|\alpha - a/q| < q^{-2}$. For $k \geqslant 1$ and $\alpha_1, \dots, \alpha_k \in \mathbf{C}$, let ϕ be the polynomial*

$$\phi(x) = \alpha x^k + \alpha_1 x^{k-1} + \cdots + \alpha_{k-1} x + \alpha_k. \tag{3.2}$$

Then we have, for all $Q \geqslant 1$,

$$\sum_{x=1}^{Q} e(\phi(x)) \ll Q^{1+\varepsilon} \left(q^{-1} + Q^{-1} + qQ^{-k}\right)^{2^{1-k}}. \tag{3.3}$$

Lemma 2. *For all $\alpha \in \mathfrak{m}$,*

$$f_2(\alpha) \ll f_2(0) n^{-\tau/2+\varepsilon}. \tag{3.4}$$

Proof. This is Weyl's inequality applied to $\phi = \alpha x^2$. We use the fact that in this case $k = 2$, $Q \asymp n^{1/2}$ and $n^\tau \leqslant q \leqslant n^{1-\tau}$ by definition of the minor arcs and Dirichlet's lemma on Diophantine approximation. □

Remark. We do not use Weyl's inequality to bound f_8 on minor arcs since the gain would be inefficiently small.

3.3 MEAN VALUE THEOREMS

In order to bound the mean squares in (3.1), recall the mean value theorems of [Vau89a, VW95] and [Woo92], as well as Ford's optimization algorithm [For95].

Lemma 3 (Vaughan-Wooley)**.** *There is a $\gamma > 0$ so that, for every integers $k \geqslant 3$ and $s \geqslant 1$, there is a computable $\lambda(k,s) > 0$ such that*

$$\int_0^1 |g_k(\alpha)|^{2s} \mathrm{d}\alpha \ll n^{\lambda(k,s)/k+\varepsilon}. \tag{3.5}$$

Explicit tables of exponents for this mean square theorem are algorithmically computable by the three methods described in [Vau89a, Theorem 4.1], [Vau89b, Lemma 2.3] and [Woo92, Lemma 3.2] optimized in the way described by Wooley [Woo92]. The implementation of the algorithm in Sage [The20] providing all the values used in this paper is available on the authors' webpages. See Table 1 below for some output of this algorithm, in particular containing Ford's tables whose algorithm is undisclosed.

k	$\lambda(k,k)$	k	$\lambda(k,k)$
4	4.60572553279363	40	50.9338839916435
6	7.31830866162191	42	53.4919522856964
8	9.92905727118400	44	56.0499163246911
10	12.5085676596728	46	58.6077897648850
12	15.0810335354744	48	61.1655839817793
14	17.6492420253841	50	63.7233085263161
16	20.2147016775680	52	66.2809714759776
18	22.7782942010074	54	68.8385797079435
20	25.3405652008671	56	71.3961391137431
22	27.9018686743506	58	73.9536547694960
24	30.4624435937399	60	76.5111310720912
26	33.0224567697859	62	79.0685718489890
28	35.5820280054141	64	81.6259804474121
30	38.1412454741396	66	84.1833598073007
32	40.7001754622901	68	86.7407169813713
34	43.2588687351309	70	89.2980408848625
36	45.8173648117595	72	91.8553469369745
38	48.3756949057251	74	94.4126324955738

TABLE 1. Values of exponents in the mean value theorem.

Note that $\lambda(k,s)$ is defined only for integer values of s in the above result. However, by Hölder's inequality we have, for any $\theta \in (0,1)$ and $k,h \geqslant 3$,

$$\int_0^1 |g_k|^{2(h+\theta)} \leqslant \left(\int_0^1 |g_k|^{2h}\right)^{1-\theta} \left(\int_0^1 |g_k|^{2(h+1)}\right)^{\theta}$$
$$\ll n^{\frac{1}{k}((1-\theta)\lambda(k,h)+\theta\lambda(k,h+1))},$$

so that the relation (3.5) still holds for any real value of $s = h+\theta$ by letting

$$\lambda(k,h+\theta) = (1-\theta)\lambda(k,h)+\theta\lambda(k,h+1), \qquad \theta \in (0,1). \tag{3.6}$$

To estimate $g_k(0)$, we use the fact that the set of smooth numbers is full-sized [For95, Lemma 3.4] in the sense that, for all $\gamma > 0$, we have $|\mathscr{A}(n^{1/k}, n^{\gamma})| \gg n^{1/k}$, so that $g_k(0)$ is of size $n^{1/k}$ for all $k \geqslant 1$. We therefore get, for $i \in \{1,2\}$ and real numbers a_k's such that $\sum_k a_k^{-1} = 1$,

$$\int_0^1 |F_i|^2 \ll \prod_{k \in K_i} \left(\int_0^1 |g_k|^{2a_k}\right)^{1/a_k} \ll F_i(0)^2 n^{\phi_i}, \qquad i \in \{1,2\}.$$

where ϕ_i is given by

$$\phi_i = \sum_{k \in K_i} \frac{\lambda(k,a_k)}{ka_k} - 2\sum_{k \in K_i} \frac{1}{k}. \tag{3.7}$$

Remark. This formalism will be steadily used all along the paper, the choices for F_1 and F_2 changing from one section to another, always referring to a partition of the set K (up to some powers already taken care of). The choice of s entirely determines the optimal a_k by the optimization algorithm described by Ford, and therefore the exponent ϕ_i. This exponent decreases with s, so that at each step of the argument we can choose the least possible s. We use Ford's heuristics [For96]: a choice close to the optimal is to take the $a_k's$ such that $a_{k_i}k_j = a_{k_j}k_i$ for all i, j.

3.4 MIXED MEAN VALUE THEOREMS

The problem is that the above mean value theorem only holds in the case of smooth functions. However, the presence of non-smooth functions f_k is necessary to apply iterative methods. We extend the above mean value theorems by allowing non-smooth functions f_h. Let $h \in K_f$ and $k_1, \dots, k_r \in K_g$. Ford [For96] provides an algorithm whose output is a ϕ such that

$$\int_0^1 |f_h g_{k_1} \cdots g_{k_r}|^2 \ll n^{\phi}, \tag{3.8}$$

with

$$\phi = \sum_{i=1}^{r} \frac{x_i}{k_i} \nu(h, k_i, 1/x_i), \tag{3.9}$$

where the $\nu(h, k_i, x)$ and the optimal values for the convex coefficients $(x_i)_i$ are algorithmically computable. As this algorithm is essential in our paper, we have implemented it. The Sage code used in this article is provided on the authors' webpages ; it can be used for any set of powers $(k_i)_i$.

3.5 TREATMENT OF MINOR ARCS

All the tools are now at hand to bound the quantities appearing in (3.1). Recall that we defined $\mathfrak{M} = \mathfrak{M}(n^\tau)$ and $\mathfrak{m} = \mathfrak{m}(n^\tau)$ where τ has to be chosen so that s is the least possible. By the Cauchy-Schwarz inequality, the Weyl inequality and the above mean value theorems, we have

$$\begin{aligned}\int_{\mathfrak{m}} F &\ll \sup_{\mathfrak{m}} |f_2| \sup_{\mathfrak{m}} |f_8| \left(\int_0^1 |F_1|^2\right)^{1/2} \left(\int_0^1 |F_2|^2\right)^{1/2} \\ &\ll f_2(0) f_8(0) n^{-\tau/2+\varepsilon} \left(F_1(0)^2 n^{\phi_1}\right)^{1/2} \left(F_2(0)^2 n^{\phi_2}\right)^{1/2} \\ &\ll F(0) n^{\frac{1}{2}(\phi_1+\phi_2-\tau)+\varepsilon}.\end{aligned}$$

Remarks. We can shed some light on the choices made for τ and s :

(i) Assume for the sake of symmetry that $\phi_1 \simeq \phi_2 \simeq \phi$. The above exponent in n has to be less than -1 for the minor arcs contribution to be negligible compared to the expected main term in Theorem 2. It is therefore necessary to have $\phi - \tau/2 < -1$. In particular, the bound improves for larger τ and smaller ϕ (*i.e.* larger s).

(ii) Larger τ allows smaller values of s. Since τ cannot be larger than $1/2$, this implies $\phi < -3/4$, providing a lower bound on s with the chosen method: $s \geqslant 72$.

(iii) However, larger τ worsen the estimates on major arcs. The final choice therefore is a trade-off between the quality of the bound on minor arcs and the quality of approximation on major arcs in Section 4, in order to get the least possible s. This is the reasoning ultimately leading to the following choices.

From now on, let $\tau = 0.3935$ and $s = 173$. We get the following :

Lemma 4. *There is $\delta > 0$ such that*

$$\int_{\mathfrak{m}} F \ll F(0) n^{-1-\delta}. \tag{3.10}$$

Proof. The mixed mean-value algorithm described above yields the values

$$\begin{aligned} \phi_1 &= -0.800823 \quad \text{with} \quad K_1 = \{6, 10, 12 \ldots, 66\}, \\ \phi_2 &= -0.805793 \quad \text{with} \quad K_2 = \{4, 68, 70, \ldots, 346\}. \end{aligned}$$

By Cauchy-Schwarz inequality we get, for $\delta = 0.000058$,

$$\begin{aligned} \int_{\mathfrak{m}} F &\ll \sup_{\mathfrak{m}} |f_2| \sup_{\mathfrak{m}} |f_8| \left(\int_0^1 |F_1|^2 \right)^{1/2} \left(\int_0^1 |F_2|^2 \right)^{1/2} \\ &\ll f_2(0) f_8(0) n^{-\tau/2+\varepsilon} \left(F_1(0)^2 n^{\phi_1} \right)^{1/2} \left(F_2(0)^2 n^{\phi_2} \right)^{1/2} \\ &\ll F(0) n^{-1-\delta}. \quad \square \end{aligned}$$

We therefore deduce the following reduction of the problem, for a certain $\delta > 0$:

$$\int_0^1 F(\alpha) e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}} F(\alpha) e(-n\alpha) \mathrm{d}\alpha + O\big(F(0) n^{-1-\delta}\big). \tag{3.11}$$

4. APPROXIMATION ON MAJOR ARCS

4.1 APPROXIMATED VERSIONS

We introduce the approximated versions of the generating functions f_k and g_k. Introduce $e_q(x) = e(x/q)$. In this whole section, all the α considered are in $\mathfrak{M}$. Note ρ the Dickman's function introduced in [For96, Section 4]. For $k \geqslant 1$, let $(a, q) = 1$ and $|\beta| < 1/2$ such that

$\alpha = \frac{a}{q} + \beta$. Define

$$\begin{aligned} S_k(q,a) &= \sum_{m=1}^{q} e_q\big(am^k\big), \\ w_k(\beta) &= \sum_{m\leqslant n} k^{-1} m^{1/k-1} e(\beta m), \qquad \text{for } k \leqslant 8, \\ w_k(\beta) &= \sum_{n^{\gamma k}<m\leqslant n} k^{-1} m^{1/k-1} \rho\left(\tfrac{\log m}{\gamma k \log n}\right) e(\beta m), \qquad \text{for } k \geqslant 9, \\ W_k(\alpha) &= q^{-1} S_k(q,a) w_k(\beta), \\ \Delta_k(\alpha) &= f_k(\alpha) - W_k(\alpha). \end{aligned}$$

We want to replace the f_k and g_k by W_k for each $k \geqslant 2$, up to an error term. To this end, recall the following bounds [For95, Lemmas 4.1 and 4.2].

Lemma 5. *Let* $1 \leqslant k \leqslant 8$*. For all* $(a,q) = 1$ *and* $|\beta| < 1/2$ *we have, for* $\alpha = \frac{a}{q} + \beta$*,*

$$W_k(\alpha) \ll \left(\frac{n}{q}\right)^{1/k} (1 + n|\beta|)^{-1}. \tag{4.1}$$

For $k \geqslant 9$*, we have*

$$W_k(\alpha) \ll \left(\frac{n}{q}\right)^{1/k} (1 + n|\beta|)^{-1/k}. \tag{4.2}$$

Lemma 6. *Let* $k \geqslant 1$*. For all* $(a,q) = 1$ *and* $|\beta| < 1/2$ *we have, for* $\alpha = \frac{a}{q} + \beta$*,*

$$\Delta_k(\alpha) \ll q^{1/2+\varepsilon} (1 + n|\beta|)^{1/2}. \tag{4.3}$$

4.2 REPLACING f_2

In order to replace f_2 by its approximated version W_2, use Lemma 6 on the major arcs $\mathfrak{M}$. This yields

$$\Delta_2 \ll n^{\tau/2+\varepsilon} \ll f_2(0) n^{(\tau-1)/2+\varepsilon}. \tag{4.4}$$

Therefore, by the mean value bounds (3.5) with the same F_1, F_2 and δ as in the previous section, we have

$$\begin{aligned} \int_{\mathfrak{M}} |\Delta_2 f_4 f_6 f_8 g_{10} F_2| &\ll f_2(0) f_8(0) n^{\frac{\tau-1}{2}+\varepsilon} \left(\int_0^1 |F_1|^2\right)^{1/2} \left(\int_0^1 |F_2|^2\right)^{1/2} \\ &\ll F(0) n^{(\tau-1)/2+\phi_1/2+\phi_2/2+\varepsilon} \\ &\ll F(0) n^{-1-\delta}. \end{aligned}$$

The problem is therefore reduced as follow, for a certain $\delta > 0$:

$$\int_{\mathfrak{M}} F(\alpha) e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}} W_2 f_4 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha + O\big(F(0) n^{-1-\delta}\big). \tag{4.5}$$

4.3 REPLACING f_4

The following pruning lemma [Brü88] is central in the approximation process.

Lemma 7 (Brüdern, pruning lemma)**.** *Let $X \leqslant n$. For $1 \leqslant a \leqslant q \leqslant X$ with $(a,q) = 1$, let $G : \mathfrak{M}(X) \to \mathbf{C}$ be a function such that*

$$G(\alpha) \ll \frac{n}{q}(1+n\beta)^{-1}, \qquad \text{for } \alpha \in \mathfrak{M}(X;q,a). \tag{4.6}$$

Let $\Psi : \mathbf{R} \to [0,+\infty)$ be a function with Fourier expansion of the form

$$\Psi(\alpha) = \sum_{|h| \ll n^{\eta}} \psi_h e(\alpha h), \tag{4.7}$$

for a certain $\eta > 0$, and such that

$$\psi_0 = \int_0^1 \Psi(\alpha) \mathrm{d}\alpha \ll X^{-1}\Psi(0). \tag{4.8}$$

Then

$$\int_{\mathfrak{M}(X)} G(\alpha)\Psi(\alpha)\mathrm{d}\alpha \ll n^{\varepsilon}\Psi(0). \tag{4.9}$$

The strategy is to apply the pruning lemma with $G = W_2^2$ and $\Psi = |F_1|^2$. The Cauchy inequality would then give

$$\begin{aligned}
\int_{\mathfrak{M}} |W_2 \Delta_4 F_1 F_2| &\ll \sup_{\mathfrak{M}} |\Delta_4| \left(\int_{\mathfrak{M}} |W_2^2 F_1^2| \right)^{1/2} \left(\int_0^1 |F_2^2| \right)^{1/2} \\
&\ll n^{\tau/2+\varepsilon} (F_1(0)^2 n^{\varepsilon})^{1/2} \left(F_2(0)^2 n^{\phi_2}\right)^{1/2} \\
&\ll F(0) n^{\frac{1}{2}(\tau - 3/2 + \phi_2) + \varepsilon}.
\end{aligned}$$

Remarks. This strategy motivates some heuristic comments :

(i) To satisfy the assumption of the pruning lemma, we need to choose K_1 such that $\phi_1 < -\tau$. Moreover, for the bound to be sufficient for our purposes, we need the final exponent to satisfy $\tau + \phi_2 < -\frac{1}{2}$. These conditions add up to an optimization problem that ultimately justifies the choices made for τ, s and the sets K_1, K_2.

(ii) Unlike the minor arcs situation, larger τ yield worse bounds since the requirement on ϕ_1 would be stronger. We chose ϕ_1 to be as close as possible to $-\tau$; and ϕ_2 is chosen to ensure negligibility.

Let

$$\begin{aligned}
K_1 &= \{8, 40, \ldots, 70\}, \\
K_2 &= \{6, 10, \ldots, 38, 72, \ldots, 346\}.
\end{aligned}$$

In order to replace f_4 by W_4, we want to use the pruning lemma with $G = W_2^2$ and Ψ defined by $|F_1|^2 = |f_8 g_{40} \cdots g_{70}|^2$. The Vaughan iterative algorithm yields

$$\int_0^1 \Psi \ll \Psi(0) n^{-0.395447} \ll \Psi(0) n^{-\tau}, \tag{4.10}$$

so that the pruning lemma applies and gives

$$\int_{\mathfrak{M}} |W_2 F_1|^2 \ll n^{\varepsilon} F_1(0)^2. \tag{4.11}$$

Moreover, with $F_2 = f_6 g_{10} \cdots g_{38} g_{72} \cdots g_{346}$, the mixed mean value theorem provides the required value in order to have a small enough bound below, namely

$$\int_0^1 |F_2|^2 \ll F_2(0)^2 n^{-0.919013}. \tag{4.12}$$

We therefore can conclude by Cauchy-Schwarz inequality and with the bound for Δ_4 given in Lemma 6. Let $\delta = 0.00075$. We have

$$\begin{aligned} \int_{\mathfrak{M}} |W_2 \Delta_4 F_1 F_2| &\ll \sup_{\mathfrak{M}} |\Delta_4| \left(\int_0^1 |W_2^2 F_1^2| \right)^{1/2} \left(\int_0^1 |F_2^2| \right)^{1/2} \\ &\ll n^{\tau/2+\varepsilon} (n^{\varepsilon} F_1(0)^2)^{1/2} \left(F_2(0)^2 n^{-0.895} \right)^{1/2} \\ &\ll F(0) n^{-1-\delta}. \end{aligned}$$

We therefore reduced the problem as follows, for a certain $\delta > 0$:

$$\int_{\mathfrak{M}} W_2 f_4 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}} W_2 W_4 f_6 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha + O\big(F(0) n^{-1-\delta}\big).$$

5. PRUNING MAJOR ARCS

5.1 PRUNING TO n^{κ}

In order to be able to replace f_6 by W_6, we have to prune the major arcs. Introduce new major arcs $\mathfrak{M}_1 = \mathfrak{M}(n^{\kappa})$ for a $\kappa < \tau$ and the associated relative minor arcs $\mathfrak{m}_1 = \mathfrak{M} \backslash \mathfrak{M}_1$. We will take $\kappa = 1/4$ for reasons that will appear in Section 5.2. We have, by Lemma 5,

$$|W_2 W_4|^2 \ll \left(\frac{n}{q} \right)^{3/2} (1 + n\beta)^{-4}. \tag{5.1}$$

Examine more precisely the elements in $\mathfrak{m}_1$. For any $0 < \theta < \kappa$, we have either $q \geqslant n^{\kappa-\theta}$ (case I) or $\beta \geqslant n^{\theta-1}$ (case II). We therefore have one of the two bounds

$$\begin{aligned} &\text{(case I)} & |W_2 W_4|^2 &\ll \frac{n^{3/2}}{q} (1 + |\beta| n)^{-1} n^{(\theta-\kappa)/2}, \\ &\text{(case II)} & |W_2 W_4|^2 &\ll \frac{n^{3/2}}{q} (1 + |\beta| n)^{-1} n^{-3\theta}. \end{aligned}$$

Taking $\theta = \kappa/7$ in order to make both bounds match we get that, on $\mathfrak{m}_1$,

$$|W_2W_4|^2 \ll \frac{n}{q}(1+|\beta|n)^{-1}n^{1/2-3\theta}. \tag{5.2}$$

In particular the pruning lemma can be applied to $G = n^{3\theta-1/2}|W_2W_4|^2$. Define $F_1 = f_8g_{40}\cdots g_{70}$. By the mean value theorems we have

$$\int_0^1 |F_1|^2 \ll F_1(0)^2n^{-0.395447} \ll F_1(0)^2n^{-\tau}. \tag{5.3}$$

Moreover, the mean value algorithm gives, with $F_2 = f_6g_{10}\cdots g_{38}g_{72}\cdots g_{346}$,

$$\int_0^1 |F_2|^2 \ll F_2(0)^2n^{-0.919013}. \tag{5.4}$$

Hence the pruning lemma can be applied with the above G and $\Psi = |F_1|^2$. Altogether, the contribution of the minor arcs are shown to be negligible, since by Cauchy-Schwarz inequality we get, letting $\delta = 0130$,

$$\begin{aligned}\int_{\mathfrak{m}_1} W_2W_4f_6f_8\cdots g_{346} &\ll n^{1/4-3\theta/2}\left(\int_{\mathfrak{m}_1} n^{3\theta-1/2}|W_2W_4F_1|^2\right)^{1/2}\left(\int_0^1 |F_2|^2\right)^{1/2}\\ &\ll F(0)n^{1/4-3\theta/2-1/2-1/4+\phi_2/2}\\ &\ll F(0)n^{-1-\delta}.\end{aligned}$$

In particular it is sufficient to study the integral formulation of the problem with the integral restricted to the new major arcs $\mathfrak{M}_1$. For a $\delta > 0$, we have

$$\int_{\mathfrak{M}} W_2W_4f_6f_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}_1} W_2W_4f_6f_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha + O\big(F(0)n^{-1-\delta}\big).$$

5.2 REPLACING f_6

Now that we are reduced to major arcs of size n^κ for $\kappa = 1/4$, it is possible to replace f_6 by W_6. We distribute W_k^k in Hölder's inequality. We will use the pruning lemma to write

$$\begin{aligned}&\int_{\mathfrak{M}_1} |W_2W_4\Delta_6f_8\cdots f_{2s}|\\ &\quad\ll \sup_{\mathfrak{M}_1}|\Delta_6|\left(\int_{\mathfrak{M}_1} |W_2^2F_1^2|\right)^{1/2}\left(\int_{\mathfrak{M}_1} |W_4^4F_2^2|\right)^{1/4}\left(\int_0^1 |F_2^2|\right)^{1/4}\\ &\quad\ll F(0)n^{\kappa/2-1/2-1/4-1/6+\phi_2/4}.\end{aligned}$$

Remark. It is because of this bound that the major arcs have been pruned above to n^κ with $\kappa = 1/4$. Indeed, assuming s is large enough for the pruning lemma's hypotheses to be satisfied, the bound can reach $F(0)n^{-1-\delta}$ if and only if $\kappa < 1/3$. Moreover, the closer we are of $1/3$, the closer ϕ_2 has to be to one, *i.e.* the larger s has to be. This justifies the choice of κ slightly away from $1/3$.

The above bound is negligible as soon as $\phi_2 < -5/6$. The pruning lemma is applicable as soon as $\phi_1, \phi_2 < -\kappa$. This is the case with

$$K_1 = \{64, \dots, 108\},$$
$$K_2 = \{8, \dots, 62, 110, \dots, 346\}.$$

With this choice we get

$$\int_0^1 |F_1|^2 \ll F_1(0)^2 n^{-0.255},$$
$$\int_0^1 |F_2|^2 \ll F_2(0)^2 n^{-0.911809}.$$

Finally, with $\delta = 0.0196$ we have:

$$\int_{\mathfrak{M}_1} |W_2 W_4 \Delta_6 f_8 \cdots g_{346}| \ll F(0) n^{-1-\delta}.$$

As such, this contribution is negligible, and the problem further reduces as follows, for a $\delta > 0$:

$$\int_{\mathfrak{M}_1} W_2 W_4 f_6 f_8 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}_1} W_2 W_4 W_6 f_8 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha + O\big(F(0) n^{-1-\delta}\big).$$

5.3 PRUNING TO n^ν

In order to be able to replace f_8 by W_8, we have to prune further the major arcs. Introduce new major arcs $\mathfrak{M}_2 = \mathfrak{M}(n^\nu)$ for a $\nu < \kappa$, and the associated relative minor arcs $\mathfrak{m}_2 = \mathfrak{M}_1 \backslash \mathfrak{M}_2$. We will take $\nu = 1/6.25$ for reasons that will appear in Section 5.4. We have, by Lemma 5,

$$|W_2 W_4 W_6|^2 \ll \left(\frac{n}{q}\right)^{11/6} (1 + n\beta)^{-7/3}. \tag{5.5}$$

Examine more precisely the elements in $\mathfrak{m}_2$. For any $0 < \theta < \nu$, we have either $q \geqslant n^{\nu-\theta}$ (case I) or $\beta \geqslant n^{\theta-1}$ (case II). We therefore have one of the two bounds

$$\text{(case I)} \qquad |W_2 W_4 W_6|^2 \ll \frac{n^{11/6}}{q} (1 + |\beta| n)^{-1} n^{5(\theta-\nu)/6},$$
$$\text{(case II)} \qquad |W_2 W_4 W_6|^2 \ll \frac{n^{11/6}}{q} (1 + |\beta| n)^{-1} n^{-4\theta/3}.$$

Taking $\theta = 5\nu/13$ in order to make both bounds match we get that, on $\mathfrak{m}_1$,

$$|W_2 W_4 W_6|^2 \ll \frac{n}{q} (1 + |\beta| n)^{-1} n^{\frac{5}{6} - \frac{20}{39}\nu}. \tag{5.6}$$

In particular the pruning lemma can be applied to $G = n^{\frac{20}{39}\nu-\frac{5}{6}}|W_2W_4W_6|^2$. Define $F_1 = f_8g_{268}\cdots g_{346}$. By the mean value theorems we have

$$\int_0^1 |F_1|^2 \ll F_1(0)^2 n^{-0.251069} \ll F_1(0)^2 n^{-\kappa}. \tag{5.7}$$

Moreover, the mean value algorithm gives, with $F_2 = g_{10}\cdots g_{266}$,

$$\int_0^1 |F_2|^2 \ll F_2(0)^2 n^{-0.920841}. \tag{5.8}$$

Hence the pruning lemma can be applied with the above G and $\Psi = |F_1|^2$. Altogether, the contribution of the minor arcs are shown to be negligible, since by Cauchy-Schwarz inequality we get, letting $\delta = 0.001446$,

$$\begin{aligned}\int_{\mathfrak{m}_2} W_2W_4W_6f_8\cdots g_{346} &\ll n^{\frac{5}{12}-\frac{10}{39}\nu}\left(\int_{\mathfrak{m}_2} n^{\frac{20}{39}\nu-\frac{5}{6}}|W_2W_4W_6F_1|^2\right)^{1/2}\left(\int_0^1 |F_2|^2\right)^{1/2}\\ &\ll F(0)n^{\frac{5}{12}-\frac{10}{39}\nu-1/2-1/4-1/6+\phi_2/2}\\ &\ll F(0)n^{-1/2-\frac{10}{39}\nu+\phi_2/2}\\ &\ll F(0)n^{-1-\delta}.\end{aligned}$$

In particular it is sufficient to study the integral restricted to the new major arcs $\mathfrak{M}_2$. For a $\delta > 0$, we have:

$$\int_{\mathfrak{M}_1} W_2W_4W_6f_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}_2} W_2W_4W_6f_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha + O\big(F(0)n^{-1-\delta}\big).$$

5.4 REPLACING f_8

Now that we are reduced to major arcs of size n^ν for $\nu = 1/6.25$, it is possible to replace f_8 by W_8. In order to efficiently apply the pruning lemma, we distribute W_k^k in Hölder's inequality, and aim at writing:

$$\begin{aligned}&\int_{\mathfrak{M}_2} |W_2W_4W_6\Delta_8\cdots g_{2s}|\\ &\qquad\ll \sup_{\mathfrak{M}_2}|\Delta_8|\left(\int_{\mathfrak{M}_2}|W_2^2F_1^2|\right)^{1/2}\left(\int_{\mathfrak{M}_2}|W_4^4F_2^2|\right)^{1/4}\left(\int_{\mathfrak{M}_2}|W_6^6F_2^2|\right)^{1/6}\left(\int_0^1|F_2^2|\right)^{1/12}\\ &\qquad\ll F(0)n^{\nu/2-1/2-1/4-1/6-1/8+\phi_2/12}\\ &\qquad\ll F(0)n^{-0.9616+\phi_2/12}\\ &\qquad\ll F(0)n^{-1-\delta}\end{aligned}$$

as soon as $\phi_2 < -0.46$. This is the case with

$$\begin{aligned}K_1 &= \{64,\ldots,108\},\\ K_2 &= \{10,\ldots,62,110,\ldots,346\}.\end{aligned}$$

With this choice we get

$$\int_0^1 |F_1|^2 \ll F_1(0)^2 n^{-0.255},$$
$$\int_0^1 |F_2|^2 \ll F_2(0)^2 n^{-0.896}.$$

Finally, we get, with $\delta = 0.036$,

$$\int_{\mathfrak{M}_2} |W_2 W_4 W_6 \Delta_8 g_{10} \cdots g_{346}| \ll F(0) n^{-1-\delta}.$$

As such, this contribution is negligible, and the problem further reduces as follows. For a $\delta > 0$:

$$\int_{\mathfrak{M}_2} W_2 W_4 W_6 f_8 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha = \int_{\mathfrak{M}_2} W_2 W_4 W_6 W_8 \cdots g_{2s} e(-n\alpha) \mathrm{d}\alpha + O\big(F(0) n^{-1-\delta}\big).$$

5.5 PRUNING TO $\log^A n$

Let $Y = \log^A n$ for a certain $A > 0$. Introduce the logarithmically-pruned major arcs $\mathfrak{M}_3 = \mathfrak{M}(Y)$, and $\mathfrak{m}_3 = \mathfrak{M}_2 \backslash \mathfrak{M}_3$ the associated related minor arcs. We begin by showing that the contribution of these new minor arcs is negligible. On $\mathfrak{m}_3$, by Lemma 5, we have $W_2^2, W_4^4, W_6^6 \ll \frac{n}{q}(1+n\beta)^{-1}$. Choosing

$$K_1 = \{10, 204, 206, \ldots, 346\},$$
$$K_2 = \{12, \ldots, 202\}.$$

the pruning lemma applies with the bounds

$$\int_0^1 |F_1|^2 \ll F_1(0)^2 n^{-0.346},$$
$$\int_0^1 |F_2|^2 \ll F_2(0)^2 n^{-0.876}.$$

Apply Hölder's inequality to write:

$$\int_{\mathfrak{m}_3} |W_2 W_4 W_6 W_8 F_1 F_2|$$
$$\ll \left(\int_{\mathfrak{m}_3} |W_2 F_1|^2\right)^{1/2} \left(\int_{\mathfrak{m}_3} |W_4^4 F_2^2|\right)^{1/4} \left(\int_{\mathfrak{m}_3} |W_6^6 F_2^2|\right)^{1/6} \left(\int_{\mathfrak{m}_3} |W_8^{12} F_2^2|\right)^{1/12}$$

We moreover have, by Lemma 5,

$$W_8^{12} \ll \left(\frac{n}{q}\right)^{3/2} (1+n|\beta|)^{-3/2}. \tag{5.9}$$

We should split cases as previously done, depending on whether q is far or not from Y. More precisely, splitting cases into $q \geqslant X^{1/2}$ or $\beta \geqslant n^{-1}X^{1/2}$ (for any $X < Y$), we get

$$W_8^{12} \ll \frac{n^{3/2}}{q}(1+n|\beta|)^{-1}X^{-1/4}. \tag{5.10}$$

In particular the pruning lemma can be applied to $n^{-1/2}X^{1/4}W_8^{12}$, so that

$$\begin{aligned}
&\int_{\mathfrak{m}_3} |W_2W_4W_6W_8F_1F_2| \\
&\ll n^{1/24}X^{-1/48}\left(\int_{\mathfrak{m}_3}|W_2F_1|^2\right)^{1/2}\left(\int_{\mathfrak{m}_3}|W_4^4F_2^2|\right)^{1/4} \\
&\qquad\qquad \times\left(\int_{\mathfrak{m}_3}|W_6^6F_2^2|\right)^{1/6}\left(\int_{\mathfrak{m}_3}|n^{-1/2}X^{1/4}W_8^{12}F_2^2|\right)^{1/12} \\
&\ll F(0)n^{-1}X^{-1/48}.
\end{aligned}$$

We therefore reduced the problem to estimating the integral on the new arcs $\mathfrak{M}_3$. For a certain $\delta > 0$:

$$\int_{\mathfrak{M}_2} W_2W_4W_6W_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}_3} W_2W_4W_6W_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha + O\big(F(0)n^{-1-\delta}\big).$$

5.6 PRUNING TO $\log^{1/4} n$

It is now necessary to prune again these major arcs to reach logarithmically sized arcs, so that all the remaining g_k's will be directly approachable without efforts by W_k. Introduce $Z = \log^{1/4} n$ and the new major and minor arcs $\mathfrak{M}_4 = \mathfrak{M}(Z)$ and $\mathfrak{m}_4 = \mathfrak{M}_3 \backslash \mathfrak{M}_4$ the associated relative minor arcs. On $\mathfrak{m}_4$, the Vaughan-Wooley bound [VW95] implies the following.

Lemma 8 (Vaughan-Wooley). *We have, for all* $\alpha \in \mathfrak{m}_4$,

$$g_k(\alpha) \ll \left(\frac{n}{q}\right)^{1/k} q^{\varepsilon}(1+n\beta)^{-1/k}. \tag{5.11}$$

In particular we get, by Lemma 5,

$$|W_2W_4W_6W_8g_{10}\cdots g_{2s}| \ll F(0)q^{-\omega}(1+n\beta)^{-\eta}, \tag{5.12}$$

where

$$\begin{aligned}
\omega &= \frac{1}{2}+\cdots+\frac{1}{2s}-\varepsilon, \\
\eta &= 4+\frac{1}{10}+\cdots+\frac{1}{2s}.
\end{aligned}$$

Now, integrating over the minor arcs $\mathfrak{m}_3$ we get

$$\begin{aligned}\int_{\mathfrak{m}_4} |W_2W_4W_6W_8\cdots g_{2s}| &\ll F(0)\left(\sum_{q\leqslant Z} q^{1-\omega}\int_{Z/nq}^{\infty}\frac{\mathrm{d}\beta}{(n\beta)^{\eta}} + \sum_{Z<q\leqslant Y} q^{1-\omega}\int_0^{Y/nq}\frac{\mathrm{d}\beta}{(1+n\beta)^{\eta}}\right)\\ &\ll F(0)\left(n^{-1}Z^{1-\eta}\sum_{q\leqslant Z} q^{\eta-\omega} + n^{-1}\sum_{Z<q\leqslant Y} q^{1-\omega}\right)\\ &\ll F(0)n^{-1}Z^{2-\omega}.\end{aligned}$$

Altogether, we get for $\rho = (\omega-2)/4 > 0$,

$$\int_{\mathfrak{m}_4} |W_2W_4W_6W_8\cdots g_{2s}| \ll F(0)n^{-1}\log^{-\rho} n, \tag{5.13}$$

which is negligible in front of the expected main term estimated in Theorem 2. In particular, it is enough to concentrate on the integral whose integration domain is restricted to $\mathfrak{M}_4$. For a $\rho > 0$:

$$\int_{\mathfrak{M}_3} W_2W_4W_6W_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}_4} W_2W_4W_6W_8\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha + O\big(F(0)n^{-1}\log^{-\rho} n\big).$$

5.7 REPLACING THE REMAINING g_k'S

On these log-sized major arcs, it is straightforward to replace g_k by W_k up to an error term as soon as $k \geqslant 5$. Indeed, we have [For96, Equation (4.21)] for all $k \geqslant 5$,

$$\Delta_k(\alpha) \ll n^{1/k}\log^{-3/4} n. \tag{5.14}$$

Using the fact that the major arcs $\mathfrak{M}_4$ are of length Z^2n^{-1}, we get the bound

$$\int_{\mathfrak{M}_4} |\Delta_k| \ll n^{1/k-1}\log^{-1/4} n \ll g_k(0)n^{-1}\log^{-1/4} n. \tag{5.15}$$

With the bounds given in Lemma 5, we deduce for any $r \geqslant 4$,

$$\int_{\mathfrak{M}_4} W_2\cdots W_{2r-2}\Delta_{2r}g_{2r+2}\cdots g_{2s} \ll F(0)n^{-1}\log^{-1/4} n.$$

We therefore deduce, adding finitely many such approximation error terms :

$$\begin{aligned}&\int_{\mathfrak{M}_4} W_2\cdots W_8 g_{10}\cdots g_{2s}e(-n\alpha)\mathrm{d}\alpha\\ &\qquad = \int_{\mathfrak{M}_4} W_2\cdots W_{2s}e(-n\alpha)\mathrm{d}\alpha + O\big(F(0)n^{-1}\log^{-1/4} n\big).\end{aligned}$$

Remark. Unlike the previous works of Thanigasalam [Tha68] or Ford [For95], all the $g_k's$ have been replaced by their approximated versions W_k, greatly improving the treatment of the singular quantities below.

5.8 COMPLETING THE ARCS

The last step before studying the expression yielding the main term is to replace the integration over each $\mathfrak{M}(Z;q,a)$ above by the integral over the whole circle. This is doable since W_k is very small outside $\mathfrak{M}(Z;q,a)$. We follow the method of [For96, (4.33)] and get that the complementary part is

$$\sum_{q\leqslant Z}\sum_{(a,q)=1}\int_{[0,1]\setminus\mathfrak{M}(Z;q,a)}|W_2\cdots W_{2s}(\alpha;q,a)| \ll F(0)\sum_{q\leqslant Z}q^{1-\omega}\int_{Z/qn}^{+\infty}\frac{\mathrm{d}\beta}{(n\beta)^{\eta}}$$
$$\ll F(0)n^{-1}\log^{-\rho} n.$$

Altogether, we proved the following, for $Z=\log^{1/4} n$ and a $\rho>0$:

$$\int_{\mathfrak{M}_4} W_2\cdots W_{2s}e(-n\alpha)\mathrm{d}\alpha$$
$$= \mathfrak{S}(n,Z)I(n)+O\big(F(0)n^{-1}\log^{-1/4} n\big),$$

where

$$\mathfrak{S}(n,Z) = \sum_{q\leqslant Z} A(n,q),$$
$$A(n,q) = \sum_{(a,q)=1} q^{-s}S_2\cdots S_{2s}(q,a)e_q(-an),$$
$$I(n) = \frac{1}{2^s s!}\sum_{(\star)} m_2^{-1/2}m_4^{-3/4}m_6^{-5/6}m_8^{-7/8}\prod_{k=9}^{s}\rho\left(\tfrac{\log m_k}{k\gamma\log n}\right)m_k^{1/k-1},$$

and the conditions $(\star)$ of the above sum are given by

$$\frac{n}{2^k} < m_k \leqslant n, \qquad \text{for } k\leqslant 8,$$
$$n^{\gamma k} < m_k \leqslant n, \qquad \text{for } k\geqslant 10,$$
$$n = m_2+m_4+\cdots+m_{2s}.$$

6. SINGULAR SERIES AND INTEGRAL

The treatment of the singular series and integral is simplified by the replacement of all the f_k and g_k by their approximated version W_k. The argument is analogous to [For96] and we briefly include the details for completeness.

Since the $\dfrac{\log m_k}{k\gamma\log n}$ are uniformly bounded, the explicit definition of $I(n)$ yields

$$I(n) \gg F(0)n^{-1}. \tag{6.1}$$

Now it remains to prove that $\mathfrak{S}(n,Z) \gg 1$. First of all, this is a convergent series since $S_k(q,a) \ll q^{1-1/k}$ by [Vau97, Theorem 4.2]. This henceforth yields

$$|\mathfrak{S}(n,Z) - \mathfrak{S}(n)| \leqslant \sum_{q>Z} |A(n,q)| \ll \sum_{q>Z} q^{1-\omega} \ll \log^{-0.18} n. \tag{6.2}$$

By [Vau97, Lemma 2.11] the function $A(n,q)$ is multiplicative in q. Therefore the study is reduced to the associated local factors, namely we can write

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q) = \prod_p \chi_p, \qquad \text{where} \quad \chi_p = \sum_{h=0}^{\infty} A\left(n,p^h\right). \tag{6.3}$$

The same bound as above therefore yields

$$|\chi_p - 1| \leqslant \sum_{h=1}^{\infty} \left|A\left(n,p^h\right)\right| \ll p^{1-\omega}. \tag{6.4}$$

With the bound $\chi_p \gg p^{-1188}$, consequence of [For95, Lemma 6.4] with $r = 173$ and $\gamma = 9$, this leads to $\mathfrak{S}(n) \gg 1$. In particular, $r_s(n) > 0$ for n large enough, achieving the proof of Theorem 2.

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