Mathematische Zeitschrift (2020) 295:129–159
https://doi.org/10.1007/s00209-019-02401-x

Mathematische Zeitschrift

Check for updates

Counting and equidistribution for quaternion algebras

Didier Lesesvre[1]

Received: 22 April 2019 / Accepted: 7 May 2019 / Published online: 11 September 2019

Abstract
We aim at studying automorphic forms of bounded analytic conductor in the totally definite quaternion algebra setting. We prove the equidistribution of the universal family with respect to an explicit and geometrically meaningful measure. It leads to answering the Sato–Tate conjectures in this case, and contains the counting law of the universal family, with a power savings error term.

Keywords Automorphic forms · Arithmetic statistics · Selberg trace formula · Plancherel equidistribution · Sato–Tate conjecture

1 Introduction

1.1 Landscape

Let F be a number field of degree d over $\mathbf{Q}$. Let $\mathbf{A}$ denote the ring of adeles of F. We consider a totally definite division quaternion algebra B over F, and write R for the places of F where B is not split. In particular, it contains all the archimedean places by the totally definite assumption, and this only happens for totally real fields F. We introduce the group of projective units $G = Z\backslash B^\times$, where Z denotes the center of $B^\times$. Let $\mathcal{A}(G)$ denote the universal family of G, that is the set of all irreducible automorphic infinite dimensional representations of $G(\mathbf{A})$. A deep understanding of $\mathcal{A}(G)$ is of fundamental importance in the theory of automorphic forms.

In order to determine its actual size and some sharper statistical properties, as densities or equidistribution, we need to truncate it for we then deal with a finite set, hence we need a suitable notion of size to do so. Turn for a moment to a more usual setting: the one of general linear groups. The universal family $\mathcal{A}(G)$ embeds, via the Jacquet–Langlands correspondence, as a subfamily of the universal family $\mathcal{A}(\mathrm{PGL}(2))$, composed of all the cuspidal automorphic representations of $\mathrm{PGL}(2)$. In the latter context, even in the broader setting of cusp forms on general linear groups, Iwaniec and Sarnak [20] have defined a good notion of size, given by the analytic conductor. It is a positive real number $c(\pi)$ defined

✉ Didier Lesesvre
lesesvre@math.cnrs.fr

1 School of Mathematics, Sun Yat-Sen University, Zhuhai Campus, Tangjiawan, Zhuhai 519082, Guangdong, China

from the functional equation satisfied by the finite part L-function $L(s,\pi)$ associated to $\pi \in \mathcal{A}(\mathrm{PGL}(2))$, which takes the form

$$L(1-s,\tilde{\pi}) = \bar{\varepsilon}_\pi X(s,\pi) L(s,\pi), \tag{1}$$

where ε_π is the root number of π. The completing factor $X(s,\pi)$ takes value ε_π at the central point $\frac{1}{2}$, and the analytic conductor is defined to be $c(\pi) = |X'(1/2,\pi)|$ following the presentation of Conrey et al. [13]. The function $X(s,\pi)$ involves the usual arithmetic conductor as well as archimedean gamma factors, so that the analytic conductor encapsulates the complexity of π. It allows to truncate the universal family of PGL(2), and hence the one of G, to a finite set [8]. The truncated universal family may then be introduced as

$$\mathcal{A}(Q) = \{\pi \in \mathcal{A}(G) \, : \, c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{2}$$

The problem of counting automorphic representations ordered by analytic conductor goes back to the work of Iwaniec and Sarnak [20]. In this article, we seek to prove certain basic properties of this family, such as determining its asymptotic growth, establishing global equidistribution with respect to a geometrically significant measure, and proving the validity of the Sato-Tate conjecture in this setting.

1.2 Analogy with the height on an algebraic variety

The counting problem admits an interesting analogy with the well-known question of counting rational points of bounded height on a smooth projective variety over a number field. The absolute Weil height is the proper notion of size in this setting and is defined by

$$h_{\mathbf{P}^n}(x) = \prod_v \max_{0 \leqslant i \leqslant n} |x_i|_v^{1/[F:\mathbf{Q}]}, \qquad x = (x_i)_{0 \leqslant i \leqslant n} \in \mathbf{P}^n(F), \tag{3}$$

where the product runs over the places of F and does not depend on the choice of homogeneous coordinates. Given any projective variety V over F endowed with a fixed embedding ι into the projective space $\mathbf{P}^n(F)$, a height function on V can be defined by pulling back the Weil height on $\mathbf{P}^n(F)$, setting

$$h_V(x) = h_{\mathbf{P}^n}(\iota(x)), \qquad x \in V. \tag{4}$$

The most natural setting for considering such generalized questions is the one of Fano varieties, where there are precise conjectures due to Batyrev, Manin [4] and Peyre [33]. On those grounds, Northcott [31] proved the finiteness of the set of points of bounded height for the projective spaces, refined by Schanuel [39] in an asymptotic counting law.

Theorem 1 (Schanuel) *For all $n \geqslant 1$, there exists $C_n > 0$ such that for any $Q \geqslant 1$,*

$$\#\left\{x \in \mathbf{P}^n(F) \, : \, h_{\mathbf{P}^n}(x) \leqslant Q\right\} = C_n Q^{n+1} + \begin{cases} O\left(Q \log Q\right) & \text{if } n = 1, \ F = \mathbf{Q}; \\ O\left(Q^{n-1/[F:\mathbf{Q}]}\right) & \text{otherwise.} \end{cases}$$

The analogy between the Schanuel theorem on counting rational points on projective spaces and the problem of counting automorphic cusp forms on $\mathrm{GL}(n)$ has been particularly stressed recently, according to Sarnak. The case of quaternion algebras can be embedded in $\mathrm{GL}(2)$ so that, following the above analogy, the notion of analytic conductor we use in our main theorem is inspired by the procedure for heights: given the by now standard notion of analytic conductor for $\mathrm{GL}(2)$, we pull it back to automorphic forms on quaternion algebras via the associated identity map between their dual groups, hence defining the notion of analytic conductor in our setting.

1.3 Counting law for the universal family

The first result of this article gives an asymptotic formula for the cardinality

$$N(Q) = \#\mathcal{A}(Q), \qquad Q \geqslant 1, \tag{5}$$

Petrow recently handled the problem in a fairly general fashion for automorphic forms on tori [32]. The case of the universal family for GL(2) is handled by Brumley and Milićević in the preprint [9]. For division quaternion algebras, the counting law is provided by the following statement.

Theorem 2 (Counting law for quaternion algebras) *There exists $C > 0$ such that for any $Q \geqslant 1$,*

$$N(Q) = CQ^2 + \begin{cases} O\left(Q^{1+\varepsilon}\right) & \text{if } F = \mathbf{Q} \text{ and } B \text{ totally definite, for all } \varepsilon > 0; \\ O\left(Q^{2-\delta_F}\right) & \text{otherwise.} \end{cases}$$

The constant $C > 0$ is defined explicitly in (8), and $\delta_F = 2(1 + [F : \mathbf{Q}])^{-1}$.

Remarks The form of this asymptotic growth appeals some comments.

(i) There is a similarity between the error term in Theorem 2 and that of the classical result of Schanuel in Theorem 1 on the number of rational points of bounded height in projective spaces. His result, when specialized to $F = \mathbf{Q}$, has an error term that picks up an additional small quantity, namely a power of log, to be compared to the Q^ε of Theorem 2.

(ii) The presence of a power savings error term in the totally definite case is noteworthy. This feature is lost in the corresponding result [9] for GL(2), where only a logarithmic savings is obtained. The reason for this difference lies in the passage from smooth to sharp counting at archimedean places, that does not occur in the totally definite setting.

(iii) The center has been removed for technical purposes and to avoid to deal with the central terms in the Selberg trace formula. All the methods are expected to carry on to a setting considering the center without considerable adaptation.

(iv) Relying on the machinery developped in [9], it is possible to generalize this result to general division quaternion algebras, with an error term only displaying a logarithmic savings. This is worked out in the author's PhD thesis [30].

The precise knowledge of the constant C unveils a lot of information, and its geometric interpretation has considerable importance as in the conjectures of Peyre. An explicit and meaningful formulation of the constant is given below, in the context of the equidistribution properties of $\mathcal{A}(G)$, and shows striking similarities with the ones computed for algebraic varieties [11].

1.4 Equidistribution of the universal family

Beyond estimating the size of the universal family lies the question of the geometric distribution of the automorphic representations of G. A good formulation of the problem is developed in the work of Sarnak, Shin and Templier [37] and is to find a measure with respect to which the universal family equidistributes, what is carried on in this section after giving a glance at the topological and measurable structure the universal family is endowed with.

Each local unitary dual group $\widehat{G}_v$ is endowed with the Fell topology and the product $\prod_v \widehat{G}_v$ is then given the product topology. Introduce the measure μ on $\prod_v \widehat{G}_v$ that assigns to every basic open set $X = \prod_v X_v$, *i.e.* where X_v is an open set of $\widehat{G}_v$ and $X_v = \widehat{G}_v$ for all but finitely many v, the positive real number

$$\mu(X) = \int_X^\star \frac{\mathrm{d}\pi}{c(\pi)^2}, \tag{6}$$

where the regularized integral is defined as

$$\zeta^\star(1) \prod_v \zeta_v(1)^{-1} \int_{X_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^2}. \tag{7}$$

Here, the conductors $c(\pi)$ and $c(\pi_v)$ are precisely defined in Sect. 2.1, and ζ_v is the local zeta function associated to F_v, the notation $\zeta^\star(1)$ stands for the residue of the Dedekind zeta function of F at 1, and $\mathrm{d}\pi_v$ is the Plancherel measure on $\widehat{G}_v$, introduced and normalized according to the convention in Sect. 2.2.

Remarks This integral is not as disturbing as it seems for the following reasons.

(i) The Plancherel measure is supported on the tempered dual; since tempered representations are generic, the conductors appearing in the integral are well-defined for the sets actually arising in what follows, see Sect. 2.1.
(ii) It is by no mean obvious that the integral (6) actually converges. It is the case and Sects. 2.3 and. 4.4 contain the explicit computations of the local factors ensuring the convergence as well as motivating the regularization.

The measure μ has finite total mass $\|\mu\|$. All the definitions are now in place to uncover the expression of the leading constant in Theorem 2, namely

$$C = \frac{1}{2}\mathrm{vol}(G(F)\backslash G(\mathbf{A}))\|\mu\|, \tag{8}$$

where the measure giving the volume of the automorphic quotient $G(F)\backslash G(\mathbf{A})$ is normalized as in Sect. 2.2. The main result is the following one.

Theorem 3 (Equidistribution for quaternion algebras) *The universal family of G equidistributes with respect to the measure μ, in the following sense. For every relatively quasi-compact open set X of $\prod_v \widehat{G}_v$ with boundary of measure zero,*

$$\frac{\#\{\pi \in \mathcal{A}(Q) : \pi \in X\}}{N(Q)} \longrightarrow \frac{\mu}{\|\mu\|}(X), \qquad \textit{as} \quad Q \to \infty. \tag{9}$$

Once this global equidistribution result stated, the Sato-Tate conjecture questions the behavior of the projections $\nu_\mathfrak{p}$ of the limit measure on the local components $\widehat{G}_\mathfrak{p}$ when the norm of $\mathfrak{p}$ grows. Let T_c be the subgroup of diagonal matrices in $\mathrm{SU}(2)$ and W the associated Weyl group. On the common ground where all the representations in the support of the Plancherel measures of $G_\mathfrak{p}$ live, given by the tempered Satake parameters space T_c/W, the Sato-Tate question acquires a precise meaning and local representations are equidistributed with respect to the half-circle measure.

Corollary 1 (Sato–Tate for quaternion algebras) *For all $\phi \in C(T_c/W)$,*

$$\int_{T_c/W} \widehat{\phi}(x)\mathrm{d}\nu_\mathfrak{p}(x) \longrightarrow \int_{T_c/W} \widehat{\phi}(x)\mathrm{d}\mu^{\mathrm{ST}}(x), \qquad \textit{as} \quad N\mathfrak{p} \longrightarrow \infty, \tag{10}$$

where $\mathrm{d}\mu^{\mathrm{ST}}$ *is the Sato-Tate measure on the half-circle, i.e.*

$$\mathrm{d}\mu^{\mathrm{ST}}(x) = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\mathrm{d}x. \tag{11}$$

Remark These results generalize to the case of all division quaternion algebras, adapting the work of [9], see [30].

1.5 Organization of this article

Section 2 is mainly devoted to introducing notations, stating the precise definition of the analytic conductor, and fixing the normalizations of measures. We recall some facts about equidistribution and spectral tools required to reduce Theorem 3 to a statement amenable to trace formula methods, among which the Sauvageot density theorem. In particular, we state a decomposition of the universal family into harmonic subfamilies obtained by fixing certain spectral data. In Sect. 3 we show that the proportion of automorphic forms we seek to estimate can be expressed as a spectral side of the Selberg trace formula, hence can be expressed in terms of orbital integrals. The main asymptotic term involved in this proportion comes from the contribution of the identity, which we evaluate in Sect. 4. Other spectral and geometric terms arise in the trace formula. The spectral ones are those coming from undesired characters: they are precisely bounded in Sect. 5. The geometric ones are those coming from the orbital integrals associated to other terms than the identity, and they are bounded in Sect. 6, opening the path to the claimed asymptotic development. The ultimate Sect. 7 builds on the known Plancherel measures in the split case in order to prove that the limit measure with respect to which the universal family equidistributes satisfies the Sato-Tate equidistribution conjecture.

2 Groundwork

We denote by v the places of F, $\mathfrak{p}$ the non-archimedian ones, and $\mathcal{O}_{\mathfrak{p}}$ the ring of integers of $F_{\mathfrak{p}}$ for a finite place $\mathfrak{p}$. The finite set R of ramification places of B determines it up to isomorphism. From now on, Latin letters q, d, m, etc. will denote usual integers, while Gothic letters $\mathfrak{q}, \mathfrak{d}, \mathfrak{m}$, etc. will denote ideals of integer rings. Many bounds stated in this paper depend on an arbitrary $\varepsilon > 0$, and the implied constants are allowed to depend on ε.

2.1 Analytic conductor

In order to make sense of the problem, we need to define precisely the notion of size we choose for representations. It is the analytic conductor, which we introduce in this section. We will work with $B^\times$ more than with G, for it lightens notations. This local convention makes no harm, for we view a representation π of $G(\mathbf{A}) = PB^\times(\mathbf{A})$ as a representation of $B^\times(\mathbf{A})$ with trivial central character. By Flath's theorem, an irreducible admissible representation of $B^\times(\mathbf{A})$ decomposes in a unique way as a restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible smooth representations where almost every component π_v is unramified. We want first to define the conductor for the local components π_v.

The Jacquet–Langlands correspondence as quoted in [18, Theorem 2] and [15, Theorem 10.1 and Eq. (10.1)] allows to reduce to the GL(2) case, and in this one only infinite-dimensional representation arise. For our purposes, it can be stated as follows.

Theorem 4 (Jacquet–Langlands Correspondence) *There is a unique bijection between the set* $\mathcal{A}_{\mathrm{sc}}(\mathrm{GL}(2))$ *of supercuspidal representations of* GL(2) *and the universal family of* $B^\times$,

$$\mathrm{JL} : \mathcal{A}_{\mathrm{sc}}(\mathrm{GL}(2)) \longleftrightarrow \mathcal{A}(B^\times), \tag{12}$$

satisfying, for every pair of elliptic elements $g \in \mathrm{PGL}(2)$ *and* $h \in B^\times$ *with same characteristic polynomials, the following relations on their central characters:*

$$\chi_\pi(g) = -\chi_{\mathrm{JL}(\pi)}(h). \tag{13}$$

Moreover, the corresponding L-functions are preserved through this map, that is to say

$$L(s, \pi) = L(s, \mathrm{JL}(\pi)). \tag{14}$$

Since the universal family excludes global characters, a representation π in it is generic. The Jacquet–Langlands correspondence preserves genericity hence, as shown on the diagram below, associates to π a generic representation $\mathrm{JL}(\pi)$ of GL(2), thus also its local components $\mathrm{JL}(\pi)_v$. These local components are also the images by the local Jacquet–Langlands correspondence $\mathrm{JL}_v(\pi_v)$ of the local components of π.

$$\begin{array}{ccc}
\pi \in \mathcal{A}(B^\times) & \xrightarrow{\quad \mathrm{JL} \quad} & \begin{array}{c}\mathrm{JL}(\pi) \in \mathcal{A}(\mathrm{GL}(2)) \\ \text{generic}\end{array} \\
\big\downarrow v & & \big\downarrow v \\
\pi_v & \xrightarrow[\text{id if } v \notin R]{\quad \mathrm{JL}_v \quad} & \begin{array}{c}\mathrm{JL}(\pi)_v \\ \text{generic}\end{array}
\end{array}$$

At split places, the local Jacquet–Langlands correspondence is the identity by the uniqueness in Theorem 4, for then $B_{\mathfrak{p}}^\times \simeq \mathrm{GL}(2, F_{\mathfrak{p}})$. The correspondence is unique, thus the local components at split places π_v are generic hence infinite-dimensional, proving the claim.

2.1.1 Non-archimedian case

For finite split places $\mathfrak{p}$, by definition $B_{\mathfrak{p}} \simeq M\left(2, F_{\mathfrak{p}}\right)$ so that $B_{\mathfrak{p}}^\times$ is isomorphic to $\mathrm{GL}(2, F_{\mathfrak{p}})$. The notion of local conductor for irreducible smooth infinite-dimensional representations of GL(2) has been introduced by Casselman [10]. Consider the sequence of compact open congruence subgroups

$$K_{0,\mathfrak{p}}\left(\mathfrak{p}^r\right) = \left\{ g \in \mathrm{GL}\left(2, \mathcal{O}_{\mathfrak{p}}\right) \, : \, g \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \bmod \mathfrak{p}^r \right\} \subseteq B_{\mathfrak{p}}^\times, \qquad r \geqslant 0. \tag{15}$$

The multiplicative and analytic conductors of an irreducible admissible infinite-dimensional representation $\pi_{\mathfrak{p}}$ of $B_{\mathfrak{p}}^\times$ with trivial central character are then respectively defined by

$$\mathfrak{c}(\pi_{\mathfrak{p}}) = \mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})} \quad \text{and} \quad c(\pi_{\mathfrak{p}}) = N\mathfrak{c}(\pi_{\mathfrak{p}}), \tag{16}$$

where

$$\mathfrak{f}(\pi_{\mathfrak{p}}) = \min\left\{ r \in \mathbf{N} \, : \, \pi_{\mathfrak{p}}^{K_{0,\mathfrak{p}}(\mathfrak{p}^r)} \neq 0 \right\}. \tag{17}$$

The existence of the conductor is guaranteed by the work of Casselman [10], who also states that the growth of the dimensions of the fixed vector spaces are given by

$$\dim \pi_{\mathfrak{p}}^{K_{0,\mathfrak{p}}\left(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})+i}\right)} = i+1, \qquad i \geqslant 0. \tag{18}$$

2.1.2 Archimedean case

The archimedian part of the conductor is introduced by Iwaniec and Sarnak [20]. It is built on the archimedean factors completing the L-functions associated to automorphic representations. The archimedean L-factors are of the form, for $v|\infty$,

$$L(s, \pi_v) = \prod_{j=1}^{2} \Gamma_v(s - \mu_{j,\pi}(v)), \tag{19}$$

where $\Gamma_v(s) = \pi^{-s/2}\Gamma(s/2)$ and the $\mu_{j,\pi}(v)$ are complex numbers. The analytic conductor is then locally defined to be, for $v \mid \infty$,

$$c_v(\pi) = \prod_{j=1}^{2} \left(1 + |\mu_{j,\pi}(v)|\right)^2. \tag{20}$$

Remark We cannot avoid, following Iwaniec and Sarnak, this archimedean part of the conductor and only consider its arithmetic component. Indeed, we aim at counting irreducible admissible infinite-dimensional representations of bounded conductor, but this family would be infinite without control of the archimedian conductor. For instance the family of modular forms of level one and arbitrary weights constitutes an infinite family of fixed arithmetic conductor: they give rise to the discrete series. Similarly, the family of Maass forms constitutes an infinite family of fixed weight.

2.1.3 Non-split case

Via the Jacquet–Langlands correspondence stated in Theorem 4, the non-split case is reduced to the already treated split one, analogously with the pullback of heights for algebraic varieties. The conductor of an irreducible admissible representation π_v of $B_v^\times$ is defined as the conductor of its Jacquet–Langlands transfer

$$c(\pi_v) = c(\mathrm{JL}(\pi_v)). \tag{21}$$

2.1.4 Characters

For now conductors have been defined only for generic representations. However, characters can arise as local components at ramified places as discussed above. Every character of $B_{\mathfrak{p}}^\times$ is a composition

$$B_{\mathfrak{p}}^\times \longrightarrow F_{\mathfrak{p}}^\times \longrightarrow \mathbf{C}, \tag{22}$$

where the first application is the reduced norm, and the second one a character of $F_{\mathfrak{p}}^\times$. In other words, every character of $B_{\mathfrak{p}}^\times$ is of the form $\chi_0 \circ N$ where χ_0 is a character of $F_{\mathfrak{p}}^\times$ and N the reduced norm on $B_{\mathfrak{p}}^\times$. In order to stay consistent, define the conductor of a local character at a ramified place as the conductor of its Jacquet–Langlands embedding in $\mathrm{GL}(2)$, which is defined based on the associated functional equation. Since the character $\chi_0 \circ N$ is sent to the twisted Steinberg representation St $\otimes \chi_0$, it follows explicitly

$$\mathfrak{c}(\chi_0 \circ N) = \begin{cases} \mathfrak{p} & \text{if } \chi_0 \text{ unramified;} \\ \mathfrak{c}(\chi_0)^2 & \text{if } \chi_0 \text{ ramified.} \end{cases} \tag{23}$$

2.1.5 Global analytic conductor

We introduce for an irreducible admissible representation of $B^\times(\mathbf{A})$ decomposed into $\pi = \otimes_v \pi_v$ its global analytic conductor

$$c(\pi) = \prod_v c(\pi_v). \tag{24}$$

This gives a well-defined notion of conductor, for the π_v are almost everywhere unramified, thus of conductor one. It extends to a definition for representations of $G(\mathbf{A})$, viewed as automorphic representations of $B^\times(\mathbf{A})$ with trivial central characters.

Remark Analogously to what happens for general linear groups, the conductor could have been defined directly from the L-functions associated to automorphic representations of quaternion algebras. These are provided by the Godement-Jacquet [16] construction and would avoid the appeal to an embedding in GL(n). The Jacquet–Langlands correspondence preserves the notion of L-function by Theorem 4 and hence also makes this notion of conductor for G compatible with the one obtained by pulling back the conductor on GL(2), thus this choice of exposition makes no harm compared to directly defining the conductor from the associated L-functions on G. Thus both choices of definition of the conductor coincide.

2.2 Normalization of measures

At the non-archimedean places, the measure taken on $G_\mathfrak{p}$ is the Haar measure $\mu_\mathfrak{p}$ normalized so that $K_\mathfrak{p} = \mathrm{PGL}(2, \mathcal{O}_\mathfrak{p})$, in the split case, or $K_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}^\times$ the units of a maximal order of $B_\mathfrak{p}$, in the non-split case, gets measure one. This normalisation is independent of the chosen maximal order [19]. For the archimedean places, we choose the Haar measure normalized so that the maximal compact subgroup gets measure one.

We now turn to the associated local dual groups. Denote $\mathcal{H}(G_v)$ the Hecke algebra of G_v, that is the algebra consisting of compactly supported complex-valued functions on G_v, locally constant at finite places, smooth at archimedian ones. Let $\mathcal{H}(G(\mathbf{A}))$ be the Hecke algebra of $G(\mathbf{A})$. It is the algebra generated by the restricted products $\phi = \prod_v \phi_v$, where ϕ_v is a function of $\mathcal{H}(G_v)$ and almost every local component $\phi_\mathfrak{p}$ is equal to $\mathbf{1}_{K_\mathfrak{p}}$. For such a function $\phi \in \mathcal{H}(G(\mathbf{A}))$, we extend the action of π to $\mathcal{H}(G(\mathbf{A}))$, $\pi(\phi)$ acting by the mean action of π over G with weight ϕ, that is to say

$$\pi(\phi) = \int_{G(\mathbf{A})} \phi(g)\pi(g)\mathrm{d}g. \tag{25}$$

This defines a Hilbert-Schmidt integral operator of trace class, thus we can define its Fourier transform by

$$\widehat{\phi}(\pi) = \mathrm{tr}\ \pi(\phi) = \mathrm{tr}\left(v \mapsto \int_G \phi(g)\pi(g)v\mathrm{d}g\right). \tag{26}$$

The unitary dual group $\widehat{G}_v$ is endowed with its usual Fell topology and Plancherel measure associated with the measure chosen on G_v: it is the unique positive Radon measure μ_v^{Pl} on $\widehat{G}_v$

such that the Plancherel inversion formula [42] of Harish–Chandra holds, *i.e.* for functions ϕ_v in the Hecke algebra $\mathcal{H}(G_v)$, we have

$$\int_{\widehat{G}_v} \widehat{\phi}_v(\pi_v) \mathrm{d}\mu_v^{\mathrm{Pl}}(\pi_v) = \phi_v(1). \tag{27}$$

From now on, integrals on $\widehat{G}_v$ will be written with the convention that $\mathrm{d}\pi_v = \mathrm{d}\mu_v^{\mathrm{Pl}}(\pi_v)$, leading to no ambiguity. On $\widehat{\Pi} = \prod_v \widehat{G}_v$ we consider the product topology and the Plancherel measure, denoted by μ^{Pl}, given by the product of the local ones. We have so far clarified the settings necessary to properly introduce the measure μ defined in (6).

Remark We are interested in the universal family, part of the automorphic dual, henceforth of $\widehat{\Pi}$ which is already endowed with natural topologies. We aim at equidistribution and density results, so we choose among topologies in order to strengthen those properties. We thus seek a quite weak topology, justifying the choice of the product topology instead of the restricted product one, used when discreteness of automorphic forms is sought.

2.3 Convergence of μ

Now that every measure is properly introduced, we come back to the convergence of the integral (6). In order to prove it, it is sufficient to prove the convergence of local integrals. Let us first consider archimedean places. We are able to estimate the integral for we know the involved Plancherel measures [27, Chap. V, Theorem 6]. The principal series representations with parameter ir have conductor $1 + r^2$. Their Plancherel measures are up to a constant $r\tanh(\pi r/2)\mathrm{d}r$ or $r\mathrm{cotanh}(\pi r/2)\mathrm{d}r$ according to the parity. The discrete series representation of parameter k has conductor $1 + k^2$ and Plancherel measure $k - 1$. Hence in all of the three cases, the local integrals converge as do the quantities

$$\int_0^\infty \frac{r\tanh(\pi r/2)}{(1+r^2)^2}, \quad \int_0^\infty \frac{r\mathrm{cotanh}(\pi r/2)}{(1+r^2)^2} \quad \text{and} \quad \sum_{k\geqslant 1} \frac{k-1}{k^4}. \tag{28}$$

As for the finite places where B splits, Sect. 4.4 computes the associated local integrals which have finite values. The regularization of the integral (6) is specifically chosen in order to make the infinite product of those values convergent. The integrals at ramified places are reduced to treating the previous case by the Jacquet–Langlands correspondence stated in Theorem 4, so also converge.

2.4 Elements of equidistribution

Let S be a finite set of places of B. Define $F(\widehat{G}_S)$ to be the space of complex bounded functions on $\widehat{G}_S$ supported on a finite number of Bernstein components and whose restriction to the tempered spectrum is continuous outside a set of measure zero for the Plancherel measure restricted to each Bernstein component. Introduce the distribution measure of the truncated universal family,

$$\mu_Q = \frac{1}{N(Q)} \sum_{\pi\in\mathcal{A}(Q)} \delta_\pi, \qquad Q \geqslant 1.$$

For a positive Radon measure ν on $\widehat{\Pi}$, let

$$\nu(f) = \int_{\widehat{\Pi}} f(\pi) \mathrm{d}\nu(\pi), \qquad f \in F(\widehat{G}_S), \tag{29}$$

We say that a sequence $(\nu_n)_n$ of positive Radon measures on $\widehat{\Pi}$ weakly converges to a measure ν if $\nu_n(f)$ converges to $\nu(f)$ for every $f \in F(\widehat{G}_S)$ when n goes to infinity, for every finite set of places S. Since the characteristic functions of relatively quasi-compact open sets of $\widehat{\Pi}$ with zero-measure boundary lie in $F(\widehat{G}_S)$ by the results of Sauvageot [38, Lemme 7.2], this proves that weak convergence of μ_Q to $\mu/\|\mu\|$ implies Theorem 3. From now on we deal with the measure

$$\nu_Q = \frac{1}{Q^2} \sum_{\pi \in \mathcal{A}(Q)} \delta_\pi, \qquad Q \geqslant 1, \tag{30}$$

easier to handle than μ_Q. This is motivated by the fact, from Theorem 2, that $N(Q)$ is of asymptotical order CQ^2, so that Theorem 3 is equivalent to: ν_Q weakly converges to the measure

$$\nu = C \frac{\mu}{\|\mu\|} = \frac{1}{2} \mathrm{vol}(G(F) \backslash G(\mathbf{A}))\mu. \tag{31}$$

2.5 The Sauvageot density theorem

In order to prove the convergence of $\nu_Q(f)$ to $\nu(f)$ for every function $f \in F(\widehat{G}_S)$, it is sufficient to prove it for Fourier transforms of functions in the Hecke algebra of G_S. Indeed, the Sauvageot density theorem [38] states that any function in $F(\widehat{G}_S)$ can be approximated in that way.

Theorem 5 (Sauvageot) *Let S be a finite set of places. For every $f \in F(\widehat{G}_S)$ and $\varepsilon > 0$, there exist functions $\phi, \psi \in \mathcal{H}(G_S)$ such that*

(i) $\forall \pi \in \widehat{G}_S,\ |f(\pi) - \widehat{\phi}(\pi)| \leqslant \widehat{\psi}(\pi)$,
(ii) $\mu_S^{\mathrm{Pl}}(\widehat{\psi}) \leqslant \varepsilon$.

Let us explain how the Sauvageot theorem allows restricting the proof of Theorem 3 only to functions that are Fourier transforms of functions in the Hecke algebra. Let $f \in F(\widehat{G}_S)$. For $\varepsilon > 0$, there exist $\phi, \psi \in \mathcal{H}(G_S)$ such that $\widehat{\phi}$ and $\widehat{\psi}$ verify the conclusions of the Sauvageot theorem. We then get

$$\begin{aligned}
\nu_Q(f) - \nu(f)| &\leqslant |\nu_Q(f) - \nu_Q(\widehat{\phi})| + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})| + |\nu(\widehat{\phi}) - \nu(f)| \\
&\leqslant \nu_Q(\widehat{\psi}) + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})| + \nu(\widehat{\psi}) \\
&\leqslant |\nu_Q(\widehat{\psi}) - \nu(\widehat{\psi})| + 2\nu(\widehat{\psi}) + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})|.
\end{aligned}$$

From the definition of ν and the domination in the Sauvageot theorem it follows, since conductors are at least one, that

$$\begin{aligned}
\nu(\widehat{\psi}) &\ll \zeta_F^\star(1) \prod_v \zeta_v(1)^{-1} \int_{\widehat{G_v}} \widehat{\psi}(\pi_v) \frac{\mathrm{d}\pi_v}{c(\pi_v)^2} \\
&\ll \prod_v \zeta_v(1)^{-1} \int_{\widehat{G_v}} \widehat{\psi}(\pi_v) \mathrm{d}\pi_v \ll \mu_S^{\mathrm{Pl}}(\widehat{\psi}) \leqslant \varepsilon
\end{aligned}$$

So that we get

$$|\nu_Q(f) - \nu(f)| \ll \varepsilon + |\nu_Q(\widehat{\psi}) - \nu(\widehat{\psi})| + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})|. \tag{32}$$

In order to prove that ν_Q weakly converges to ν, it is then sufficient to show that the second and third terms vanish for $Q \to \infty$, *i.e.* to prove the theorem for the narrower class of functions $\widehat{\phi}$ and $\widehat{\psi}$. We prove indeed slightly better than what is needed for Theorem 3, with a precise asymptotic development in the case of Fourier transforms.

Theorem 6 *For every finite set of places S and $\phi \in \mathcal{H}(\widehat{G}_S)$, and every $\varepsilon > 0$,*

$$\nu_Q\left(\widehat{\phi}\right) = \nu\left(\widehat{\phi}\right) + \begin{cases} O\left(Q^{-1+\varepsilon}\right) & \text{if } F = \mathbf{Q} \text{ and } B \text{ totally definite;} \\ O\left(Q^{-\delta_F}\right) & \text{otherwise.} \end{cases} \tag{33}$$

2.6 Sieving the universal family

In order to address the problem of the weak convergence of ν_Q to prove Theorem 6, it is necessary to decompose the universal family into smaller sets with fixed spectral data, amenable to trace formula methods. Let S be a finite set of places and $\phi \in \mathcal{H}(G_S)$. The conductor of $\pi \in \mathcal{A}(G)$ splits into local conductors, in particular can be written

$$c(\pi) = c(\pi_R)c\left(\pi_S^R\right) N\mathfrak{c}\left(\pi^{R,S}\right). \tag{34}$$

This decomposition emphasizes the different kind of information and behavior each type of place is endowed with, and turns to be a guide for the method. Given a finite set of places S, recall that every ideal $\mathfrak{m}$ is decomposed in the form $\mathfrak{m} = \mathfrak{m}_S\mathfrak{m}^S$, where such a decomposition always means that $\mathfrak{m}^S$ is the prime-to-S part of $\mathfrak{m}$, *i.e.* is such that $\mathfrak{m}^S \wedge S = 1$, and $\mathfrak{m}_S$ if the S-part of $\mathfrak{m}$, *i.e.* satisfies $\mathrm{supp}(\mathfrak{m}_S) \subseteq S$. The same decomposition is used without further notice for the other letters. The multiplicative conductor of the finite split places is fixed to a certain ideal $\mathfrak{q}$ of $\mathcal{O}^R$, and the isomorphism class of the ramified part is fixed to a certain isomorphism class $\sigma_R \in \widehat{G}_R$. Thus, the universal family $\mathcal{A}(Q)$ decomposes as

$$\mathcal{A}(Q) = \bigsqcup_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R = 1}} \bigsqcup_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q/N\mathfrak{q}}} \mathcal{A}(\mathfrak{q}, \sigma_R), \tag{35}$$

where the notation $\mathfrak{q} \wedge R = 1$ stands for the fact that no place of R divides $\mathfrak{q}$, and where the sets of fixed spectral data are

$$\mathcal{A}(\mathfrak{q}, \sigma_R) = \left\{\pi \in \mathcal{A}(G) \ : \ \pi_R \simeq \sigma_R, \ \mathfrak{c}(\pi_f^R) = \mathfrak{q}\right\}.$$

This decomposition (35) of the universal family reduces the study of the whole family to the harmonic families $\mathcal{A}(\mathfrak{q}, \sigma_R)$, easier to grasp in the context of trace formulas. What is critical is to having got rid of the condition of belonging to $\mathcal{A}(Q)$, decomposed into local conditions. It induces a decomposition of the counting measure as

$$
\begin{aligned}
\nu_Q(\widehat{\phi}) &= \frac{1}{Q^2} \sum_{\pi \in \mathcal{A}(Q)} \widehat{\phi}(\pi) \\
&= \frac{1}{Q^2} \sum_{\substack{\pi \in \mathcal{A}(G) \\ c(\pi_R) c\left(\pi_f^{R,S}\right) N\mathfrak{c}\left(\pi_S^R\right) \leqslant Q}} \widehat{\phi}(\pi) \\
&= \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\pi \in \mathcal{A}(\mathfrak{q}, \sigma_R)} \widehat{\phi}(\pi)
\end{aligned}
\tag{36}
$$

where the sum over $\mathfrak{q}$ is meant to run through ideals of $\mathcal{O}^R$. Denote $A(\mathfrak{q}, \sigma_R; \phi)$ the innermost part of the splitting, that is to say

$$
A(\mathfrak{q}, \sigma_R; \phi) = \sum_{\pi \in \mathcal{A}(\mathfrak{q}, \sigma_R)} \widehat{\phi}(\pi). \tag{37}
$$

2.7 Old and new forms

The universal family (2) sees no multiplicities by the multiplicity one theorem for GL(2), but the trace formula counts them. The spectral multiplicities associated to the spectral decomposition of $L^2(G(F)\backslash G(\mathbf{A}))$, which are more suitable weights for the forthcoming computations, are given by

$$
m(\pi, \mathfrak{q}) = \dim \left(\pi^{\overline{K}_0(\mathfrak{q})}\right), \tag{38}
$$

where

$$
ZK_0(\mathfrak{q}) = \prod_{\mathfrak{p}^r || \mathfrak{q}} Z_{\mathfrak{p}} K_{0,\mathfrak{p}}\left(\mathfrak{p}^r\right) \subseteq B^\times\left(\mathbf{A}_f^R\right), \tag{39}
$$

and $\overline{K}_0(\mathfrak{q})$ stands for the image of $ZK_0(\mathfrak{q})$ under the natural projection $B^\times \to G$. The choice is made so that $m(\pi, \mathfrak{q}) \neq 0$ is equivalent to $\mathfrak{c}(\pi_f^R) \mid \mathfrak{q}$. The analogous sum to (37) additionally weighted by the multiplicities is

$$
B(\mathfrak{q}, \sigma_R; \phi) = \sum_{\pi \in \mathcal{B}(\mathfrak{q}, \sigma_R)} m\left(\pi^S, \mathfrak{q}^S\right) \widehat{\phi}(\pi), \tag{40}
$$

where

$$
\mathcal{B}(\mathfrak{q}, \sigma_R) = \left\{\pi \in \mathcal{A}(Q) \ : \ \pi_R \simeq \sigma_R, \ \mathfrak{c}\left(\pi_f^R\right) \mid \mathfrak{q}\right\}.
$$

The sum defined by (37) counts the newforms while (40) counts the old ones with respect to finite prime-to-S split places. The relation between them lies in the following lemma.

Lemma 1 *Let $\mathfrak{q}$ prime to R, σ_R an irreducible unitary representation of G_R and $\phi \in \mathcal{H}(G_S)$. Let $\lambda_2 = \mu \star \mu$ where μ is the Möbius function. For every $Q \geqslant 1$,*

$$
A(\mathfrak{q}, \sigma_R; \phi) = \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B(\mathfrak{d}, \sigma_R; \phi)
$$

Proof Recall that, for every finite split place $\mathfrak{p}$, Casselman gives the local multiplicites

$$
\dim \sigma_{\mathfrak{p}}^{K_0\left(\mathfrak{p}^{\mathfrak{f}(\sigma_{\mathfrak{p}})+i}\right)} = i + 1, \qquad i \geqslant 0. \tag{41}
$$

From this immediately follows, after taking the product over all finite split places, that the global multiplicities are

$$m\left(\sigma, \mathfrak{q}\right) = \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{c}\left(\sigma_f^R\right)}\right), \tag{42}$$

where $\tau_2 = 1\star 1$ is the divisor function. Since $\left(\sigma^R\right)^{\overline{K}_0(\mathfrak{q})} \neq 0$ implies $\mathfrak{c}(\sigma^R) \mid \mathfrak{q}$, the sum defining $B(\mathfrak{q}, \sigma_R; \phi)$ is eventually reduced to a sum over $\mathfrak{c}(\sigma^R) \mid \mathfrak{q}$. Thus, by the precise knowledge (42) of the multiplicities,

$$\begin{aligned} B\left(\mathfrak{q}, \sigma_R; \phi\right) &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \sum_{\sigma \in \mathcal{A}(\mathfrak{d}, \sigma_R)} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{c}\left(\sigma_f^R\right)}\right) \widehat{\phi}(\sigma) \\ &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) \sum_{\sigma \in \mathcal{A}(\mathfrak{d}, \sigma_R)} \widehat{\phi}(\sigma) \\ &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) A\left(\mathfrak{d}, \sigma_R; \phi\right) \end{aligned} \tag{43}$$

so that $B = \tau_2 \star A$, with a slight abuse of notation. Hence, by Möbius inversion,

$$A\left(\mathfrak{q}, \sigma_R; \phi\right) = \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B\left(\mathfrak{d}, \sigma_R; \phi\right), \tag{44}$$

achieving the proof of the claim. □

Summing over the spectral data appearing in the decomposition (36), the counting measure rewrites as

$$\nu_Q w\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R)) \\ \mathfrak{q} \wedge R = 1}} \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B\left(\mathfrak{d}, \sigma_R; \phi\right). \tag{45}$$

3 Trace formula

Trace formulæ give relations between spectral and geometric quantities, the latter being often easier to manipulate. We present here the Selberg trace formula [3, Eq. (3.4)] and express the sought old forms numbers $B(\mathfrak{d}, \sigma_R; \phi)$ as a spectral side of this trace formula for a suitable test function, leaving us with the geometric side to estimate.

3.1 Selberg trace formula

Since the automorphic quotient of G is compact, the original formulation of the trace formula, due to Selberg, can be used and combined with the multiplicity one theorem. If Φ is a function in the Hecke algebra $\mathcal{H}(G(\mathbf{A}))$, then

$$J_{\text{geom}}(\Phi) = J_{\text{spec}}(\Phi), \tag{46}$$

where the spectral and geometric parts are as follows. For $\gamma \in G$, let G_γ be the stabilizer of γ in G. The geometric part is defined by

$$J_{\mathrm{geom}}(\Phi) := \sum_{\{\gamma\}} \mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x, \tag{47}$$

where the sum runs through conjugacy classes $\{\gamma\}$ in $G(F)$. Since Φ is compactly supported and $G(F)$ is discrete in $G(\mathbb{A})$, the sum is finite. However its length depends on the support of Φ what turns to be a critical difficulty for estimations, for this support will depend on the spectral parameters. The inner integrals appearing in this geometric side are called the orbital integrals, $\mathcal{O}_\gamma(\Phi)$, defined by

$$\mathcal{O}_\gamma(\Phi) = \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x. \tag{48}$$

The spectral part is

$$J_{\mathrm{spec}}(\Phi) = \sum_{\pi \subseteq L^2(G(F)\backslash G(\mathbf{A}))} m(\pi)\widehat{\Phi}(\pi). \tag{49}$$

Here π goes through the isomorphism classes of unitary irreducible subrepresentations of $G(\mathbf{A})$ in $L^2(G(F)\backslash G(\mathbf{A}))$, and recall that $\widehat{\Phi}$ is the Fourier transform of Φ, see Sect. 2.2.

Remark The formulaton of the spectral part (49) is Selberg's original one. The weights $m(\pi)$ are the multiplicities of the π's in the discrete part of the spectral decomposition of $L^2(G(F)\backslash G(\mathbf{A}))$. The multiplicity one theorem ensures these to be less than one, and the indexation by π actually part of $L^2(G(F)\backslash G(\mathbf{A}))$ makes them nonzero, hence equal to one.

As announced in the outlook of the method, in order to have a problem amenable to the trace formula it is necessary to formulate statistics quantities on the universal family as a spectral side, hence needed to select it by the Fourier transforms of suitable test functions. The aim of the present section is to construct a function $\Phi \in \mathcal{H}(G)$ such that, up to an error term,

$$J(\Phi) = B\left(\mathfrak{d}, \sigma_R; \phi\right). \tag{50}$$

In the case of factorizable test functions $\Phi = \otimes_v \Phi_v$, the spectral side of the trace formula factorizes as

$$\widehat{\Phi}(\pi) = \prod_v \widehat{\Phi}_v(\pi_v). \tag{51}$$

Hence, in order to achieve the spectral selection (50) it is sufficient to select locally the conditions appearing in the decomposition of the universal family (45) through Fourier transforms. The following sections are dedicated to construct local test functions doing so, aim reached in Lemma 3. The places of F fall into three categories:

- the split finite part, corresponding to $\mathfrak{p} \notin R \cup S$, where the arithmetic conductor is caught by the means of an explicit filtration, see Sect. 15;
- the split finite part in the support of the test function $\widehat{\phi}$, corresponding to $\mathfrak{p} \in S\backslash R$;
- the ramified part, corresponding to the finite number of $v \in R$, which is handled by fixing the representations at those places by means of matrix coefficients.

3.2 Selecting the split conductor

For an ideal $\mathfrak{d}$ of $\mathcal{O}$, introduce the congruence subgroup given by the product of the corresponding local congruence subgroups in (15), that is to say

$$K_0(\mathfrak{d}) = \prod_{\mathfrak{p}^r \| \mathfrak{d}} K_{0,\mathfrak{p}}(\mathfrak{p}^r). \tag{52}$$

The following result gives a test function whose Fourier transform selects the finite split conductor.

Lemma 2 *For an ideal $\mathfrak{d}$ of $\mathcal{O}$, let*

$$\varepsilon_{\mathfrak{d}} = \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \mathbf{1}_{\overline{K}_0(\mathfrak{d})}. \tag{53}$$

Its Fourier transform selects the multiplicity relative to $\mathfrak{d}$. More precisely,

$$\widehat{\varepsilon_{\mathfrak{d}}}(\pi) = m(\pi, \mathfrak{d}), \qquad \pi \in \mathcal{A}(G). \tag{54}$$

Proof Let π be an automorphic representation of G. Then $\pi(\varepsilon_{\mathfrak{d}})$ is the projection of the representation space V_π on the subspace $\pi^{\mathfrak{d}}$ of the fixed vectors by $\overline{K}_0(\mathfrak{d})$ under the action of π. Indeed, every $\pi(\varepsilon_{\mathfrak{d}})v$, for v in V_π, is $\overline{K}_0(\mathfrak{d})$-invariant, for it is an averaging over the action of $\overline{K}_0(\mathfrak{d})$. For $k_0 \in \overline{K}_0(\mathfrak{d})$ and $v \in V_\pi$,

$$\begin{aligned}
\pi(k_0)\pi\left(\varepsilon_{\mathfrak{d}}\right) v &= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \pi(k_0) \int_{\overline{K}_0(\mathfrak{d})} \pi(k) v \mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k_0 k) v \mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k) v \mathrm{d}k = \pi\left(\varepsilon_{\mathfrak{d}}\right) v
\end{aligned}$$

so that its image lies in $\pi^{\mathfrak{d}}$. The action of $\pi(\varepsilon_{\mathfrak{d}})$ is also idempotent, more precisely the identity on $\pi^{\mathfrak{d}}$. Indeed, for $v_0 \in \pi^{\mathfrak{d}}$,

$$\begin{aligned}
\pi\left(\varepsilon_{\mathfrak{d}}\right) v_0 &= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k) v_0 \mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} v_0 \mathrm{d}k \\
&= v_0.
\end{aligned}$$

Hence $\pi(\varepsilon_{\mathfrak{d}})$ is an idempotent endomorphism of image $\pi^{\mathfrak{d}}$, *i.e.* a projection on $\pi^{\mathfrak{d}}$. The trace of a projection is its rank, that is to say $\widehat{\varepsilon_{\mathfrak{d}}}(\pi)$ is the dimension of the fixed vector spaces $\pi^{\mathfrak{d}}$. Those are the sought multiplicities $m(\pi, \mathfrak{d})$, in particular are nonzero if and only if $\mathfrak{c}(\pi) \mid \mathfrak{d}$. □

3.3 Selecting the ramified part

For ramified places, less is known concerning the representations and the choice made in the decomposition (45) is to fix the corresponding isomorphism class. In the finite dimensional case, knowing matrix coefficients is sufficient to determine the underlying matrix. This property still holds [25, Corollary 10.26] for supercuspidal representations in the following sense. Let σ_R be a unitary representation of G_R. A matrix coefficient, ξ_σ associated to σ_R is a function of the form, given v and w in the space of σ_R,

$$\begin{aligned} \xi_{\sigma_R}^{v,w} : G_R &\longrightarrow \mathbf{C} \\ g &\longmapsto \langle \sigma(g)v, w\rangle \end{aligned} \tag{55}$$

Matrix coefficients are continuous functions on G_R, are compactly supported since G_R is compact, and are locally constant at finite places and smooth at archimedean places.

Remark The fact that matrix coefficients is considered only for ramified places is critical for selecting purposes. The loss of the compactness of the support for matrix coefficients in the split case, where some automorphic representations are not supercuspidal, make them fail to select the corresponding isomorphism class. Such a purpose can be achieved by means of existence theorems, yet are less precise, see [25, Remark on page 214].

Proposition 1 *Let σ and π be automorphic representations of G_R, and introduce d_π the formal degree of π. Then for every pair of unit vectors v and w in the representation space of σ,*

$$\pi\left(\xi_\sigma^{v,w}\right) w = \mathbf{1}_{\pi\simeq\sigma} \frac{\langle w, v\rangle}{d_\pi} v. \tag{56}$$

Taking for v a vector of norm $d_\pi^{1/2}$, it follows that $\pi\left(\xi_\sigma^{v,v}\right)$ is the orthogonal projection onto $\mathbf{C}v$ and in the meanwhile selects the π's isomorphic to σ. Considering its trace, this can be restated as follows.

Proposition 2 *Let σ and π be automorphic representations of G_R. Let v be a vector of norm one in the representation space of σ. Then,*

$$\widehat{\xi_\sigma^{v,v}}(\pi) = \mathbf{1}_{\pi\simeq\sigma}. \tag{57}$$

From now on, denote ξ_σ any choice of matrix coefficient as in Proposition 2.

3.4 The chosen test function

The weighted counting number $B(\mathfrak{d}, \sigma_R; \phi)$ should be written as a spectral side in the trace formula. Introduce the test function

$$\Phi_{\mathfrak{d},\pi_R;\phi} = \prod_v \Phi_v, \tag{58}$$

which is built with the following local functions:

Places v	Local test function Φ_v
$\notin S, \notin R, < \infty$	$\varepsilon_{\mathfrak{d},v}$
$\notin S, \in R$	ξ_{π_v}
$\in S, \notin R, < \infty$	ϕ_v
$\in S, \in R$	$\xi_{\pi_v}\widehat{\phi_v}(\pi_v)$

where

- ϕ_v is the local component of ϕ on G_v;

- ξ_{π_v} is a matrix coefficient for π_v;
- $\varepsilon_{\mathfrak{d}}$ is the function introduced in Lemma 2, $\varepsilon_{\mathfrak{d},v}$ its v-component.

The sought weighted measure is barely reached by the spectral side with $\Phi_{\mathfrak{d},\pi_R;\phi}$, as stated in the following lemma.

Lemma 3 *Let $Q \geqslant 1$. Let $\mathfrak{d} \wedge R = 1$ and $\pi_R \in \widehat{G}_R$. Then*

$$B\left(\mathfrak{d}, \pi_R; \phi\right) = J\left(\Phi_{\mathfrak{d},\pi_R;\phi}\right) + O(\Xi(\phi, \pi_R)), \tag{59}$$

where, introducing the set $X^{\mathrm{ur}}(G)$ of unramified characters of $G(\mathbf{A})$,

$$\Xi(\phi, \pi_R) = \sum_{\substack{\chi \in X^{\mathrm{ur}}(G) \\ \chi_R \simeq \pi_R}} m(\chi^R, \mathfrak{d})\widehat{\phi}(\chi). \tag{60}$$

Proof Let $\Phi = \Phi_{\mathfrak{d},\pi_R;\phi}$. In order to determine the Fourier transform of Φ recall that for every pair of places v, w and every $a \in \mathcal{H}(G_{v,w})$, $\widehat{a_v a_w} = \widehat{a}_v \widehat{a_w}$. Thus,

$$\widehat{\Phi} = \prod_v \widehat{\Phi}_v = \prod_{v \in R} \widehat{\xi}_{\pi_v} \prod_{\substack{\mathfrak{p} \notin R \\ \mathfrak{p} \notin S \\ \mathfrak{p}^r || \mathfrak{d}}} \widehat{\varepsilon}_{\mathfrak{p}^r,v} \prod_{\substack{\mathfrak{p} \notin R \\ \mathfrak{p} \in S}} \widehat{\phi}_{\mathfrak{p}}. \tag{61}$$

Hence only the Fourier transforms of the local components of the test function have to be determined. The finite prime-to-S split part $\varepsilon_{\mathfrak{d}}$ is shown to transform into the characteristic function of conductors dividing $\mathfrak{d}$ in Lemma 2 weighted by the corresponding multiplicities. The ramified local parts ξ_{π_v} are known to transform into the characteristic functions of the isomorphism class of π_v by Proposition 2. The action of the Fourier transform of Φ follows, and (61) yields, for $\sigma \in \mathcal{A}(G)$,

$$\widehat{\Phi}(\sigma) = m(\sigma^R, \mathfrak{d})\widehat{\phi}(\sigma_f)\mathbf{1}_{\substack{\sigma_R \simeq \pi_R \\ \mathfrak{c}(\sigma^R) \mid \mathfrak{d}}}. \tag{62}$$

Nevertheless, these conditions also stand for characters: in order to not being killed by $\widehat{\Phi}$ they have to be trivial on $\overline{K}_0(\mathfrak{d})$, *i.e.* they have to be unramified since $\det(\overline{K}_0(\mathfrak{d})) = \mathcal{O}^R$. Moreover, they have to be isomorphic to π_R at ramified places. The Fourier transform of the chosen test function hence does not vanish on unramified characters, unlike awaited. The corresponding extra contribution Ξ is treated separately in Lemma 7, for characters are easier to embrace and it will be shown to contribute as an error term. □

3.5 Towards the geometric side

The equidistribution property has been recast as a convergence of spectral measures in Theorem 6. The Selberg trace formula restates it as a geometric quantity. Summing the expressions above over all the spectral data, we get

$$\begin{aligned}
\nu_Q(\widehat{\phi}) =& \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\mathrm{geom}}\left(\Phi_{\mathfrak{d},\pi_R;\phi}\right) \\
&+ O\left(\frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\sigma_R, \phi)\right).
\end{aligned}$$

The main contribution is carried by the first term, the remaining ones being showed below to contribute as negligible terms. Decompose the geometric side $J_{\text{geom}}(\Phi)$ as sum of two terms, the first one corresponding to the identity contribution, and the other being the elliptic remainder, in other words

$$J_{\text{geom}}(\Phi) = \text{vol}\,(G\,(F)\,\backslash G\,(\mathbf{A}))\,\Phi(1) + J_{\text{ell}}(\Phi), \tag{63}$$

where the elliptic part is expressed in term of orbital integrals

$$J_{\text{ell}}\,(\Phi) = \sum_{\{\gamma\}\neq\{1\}} \text{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x.$$

The universal family counting measure now decomposes, via the splitting above, as

$$\nu_Q = \text{vol}\,(G\,(F)\,\backslash G\,(\mathbf{A}))\,\nu_{1,Q} + \nu_{\text{ell}} + O(\nu_{\Xi,Q}), \tag{64}$$

where

$$\nu_{1,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Phi_{\mathfrak{d},\pi_R;\phi}(1),$$

$$\nu_{\text{ell},Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{ell}}(\Phi_{\mathfrak{d},\pi_R;\phi}),$$

$$\nu_{\Xi,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\sigma_R,\phi).$$

4 Identity contribution

For a given $\phi \in \mathcal{H}(G_S)$, the main term of $\nu_Q(\widehat{\phi})$ is given by the contribution $\nu_{1,Q}(\widehat{\phi})$ of the identity, and the other terms will be shown to be negligible. This section is dedicated to the computation of this identity contribution.

Proposition 3 *The contribution of the identity is, for $\phi \in \mathcal{H}(G_S)$,*

$$\text{vol}\,(G\,(F)\,\backslash G\,(\mathbf{A}))\,\nu_{1,Q}\left(\widehat{\phi}\right) = \nu\left(\widehat{\phi}\right) + \begin{cases} O(Q^{-1}\log Q) & \textit{if } B \textit{ is totally definite and } F=\mathbf{Q};\\ O(Q^{-\delta_F}) & \textit{otherwise.}\end{cases}$$

In particular, $\text{vol}\,(G\,(F)\,\backslash G\,(\mathbf{A}))\,\nu_{1,Q}$ *equidistributes with respect to ν.*

4.1 Evaluating the test function at 1

Before summing over the spectral data, it is necessary to look at the inner part of $\nu_{1,Q}\left(\widehat{\phi}\right)$. Fix $\mathfrak{d}$ an ideal of $\mathcal{O}^R$, π_R a unitary irreducible representation of G_R, and let for this section $\Phi = \Phi_{\mathfrak{d},\pi_R;\phi}$. The very definition (58) of Φ gives

$$\Phi(1) = \varepsilon_{K_0(\mathfrak{d}^S)}(1)\phi_S^R(1)\xi_{\sigma_R}(1)\widehat{\phi}_R(\pi_R). \tag{65}$$

4.1.1 Finite split places out of *S*

For the prime-to-S split finite part, by definition

$$\varepsilon_{\overline{K}_0(\mathfrak{d}^S)}(1) = \operatorname{vol}\left(\overline{K}_0(\mathfrak{d}^S)\right)^{-1}. \tag{66}$$

The volume of a cofinite subgroup depends on its index, and the indices of classical congruence subgroups are well-known [14]. Introduce $K^{R,S} = \prod_{v \notin R \cup S} K_v$. Since $Z^{R,S}$ is fully contained in $K_0^{R,S}(\mathfrak{d}^S)$ for all ideal $\mathfrak{d}^S$,

$$\left[\overline{K}^{R,S} : \overline{K}_0\left(\mathfrak{d}^S\right)\right] = \left[K^{R,S} : K_0\left(\mathfrak{d}^S\right)\right], \tag{67}$$

by the isomorphism theorems. So thanks to the normalizations chosen for the measures,

$$\varepsilon_{\overline{K}_0(\mathfrak{d}^S)}(1) = \left[K^{R,S} : K_0\left(\mathfrak{d}^S\right)\right] = \left(\mathrm{id}\star\mu^2\right)(\mathfrak{d}^S) =: \varphi_2(\mathfrak{d}^S). \tag{68}$$

4.1.2 Finite split places in *S*

For the S-split finite part, the Plancherel inversion formula (27) gives

$$\phi^R_{S,f}(1) = \int_{\widehat{G}^R_{S,f}} \widehat{\phi}^R_{S,f}\left(\pi^R_{S,f}\right) \mathrm{d}\pi^R_{S,f}. \tag{69}$$

4.1.3 Ramified places

For the ramified matrix coefficient (57), by the Plancherel formula (27) and the normalization chosen for ξ_{π_R},

$$\xi_{\pi_R}(1) = \int_{\widehat{G}_R} \mathbf{1}_{\sigma \simeq \pi_R} \mathrm{d}\mu^{\mathrm{Pl}}_R(\sigma) = \mu^{\mathrm{Pl}}_R(\pi_R). \tag{70}$$

4.2 Splitting the identity contribution

The following decomposition holds for the identity part of the counting measure.

Proposition 4 *For every $Q \geqslant 1$,*

$$\nu_{1,Q} = \nu^{(p)}_{1,Q} + \nu^{(e)}_{1,Q}, \tag{71}$$

where $\nu^{(p)}_{1,Q}$ is the main identity term, namely

$$\begin{aligned}\nu^{(p)}_{1,Q}(\widehat{\phi}) &= \frac{1}{2}\frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} \int_{\pi^R_S \in \widehat{G}^R_S} \frac{\widehat{\phi}(\pi^R_S)}{c(\pi^R_S)^2} \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi^R_S) \\ \mathfrak{m}^S \wedge R = 1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^2} \\ &\times \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi^R_S)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2} \mu^{\mathrm{Pl}}_R(\pi_R) \mathrm{d}\pi^R_S\end{aligned} \tag{72}$$

and where $\nu_{1,Q}^{(e)}(\widehat{\phi})$ *is an extra error term, given by*

$$\nu_{1,Q}^{(e)}(\widehat{\phi}) \ll Q^{-\delta_F+\varepsilon_F} \int_{\pi_S^R \in \widehat{G}_S^R} \frac{\widehat{\phi}(\pi_S^R)}{c(\pi_S^R)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ m^S \wedge R = 1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^{2-\delta_F+\varepsilon_F}} \mu_R^{\mathrm{Pl}}(\pi_R) \mathrm{d}\pi_S^R. \tag{73}$$

Proof The counting measure has been decomposed in measures on harmonic subfamilies (45) of fixed spectral parameters. These measures have been given a geometric interpretation by the mean of the trace formula in Lemma 3, whose identity contribution is given above. After summation of the identity contributions over the spectral data constituting the truncated universal family,

$$\nu_{1,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \int_{\pi_S^R \in \widehat{G}_S^R} \widehat{\phi}(\pi_S^R) \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\pi_S^R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \varphi_2(\mathfrak{d}^S) \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{d}c(\pi_S^R)}} \mu_R^{\mathrm{Pl}}(\pi_R)\widehat{\phi}(\pi_R) \mathrm{d}\pi_S^R.$$

Sums of arithmetic functions on ideals of number fields can be explicitly evaluated. This motivates a permutation of sums and integrals in order to estimate the sum over the volumes $\varphi_2(\mathfrak{d}^S)$ first, so that

$$\nu_{1,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \int_{\pi_S^R \in \widehat{G}_S^R} \widehat{\phi}(\pi_S^R) \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_{S,f}^R) \\ \mathfrak{m}^S \wedge R = 1}} \lambda_2(\mathfrak{m}^S) \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}}} \mu_R^{\mathrm{Pl}}(\pi_R)\widehat{\phi}(\pi_R) \sum_{\substack{N\mathfrak{d}^S \leqslant Q/N\mathfrak{m}c(\pi_R)c(\pi_{\delta,\nu}) \\ \mathfrak{d} \wedge R = 1}} \varphi_2(\mathfrak{d}^S) \mathrm{d}\pi_S^R.$$

The following lemma estimates the innermost sum.

Lemma 4 *Let* $\zeta^{S,R}$ *be the prime-to-R-and-S part of the zeta function associated to* F, *and* $\zeta^{S,R\star}(1)$ *its residue at* 1. *For any* $X > 0$,

$$\sum_{\substack{N\mathfrak{d}^S \leqslant X \\ \mathfrak{d} \wedge R = 1}} \varphi_2(\mathfrak{d}^S) = \frac{1}{2} \frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} X^2 + \begin{cases} O(X \log X) & \text{if } F = \mathbf{Q}; \\ O(X^{2-\delta_F}) & \text{otherwise.} \end{cases} \tag{74}$$

Remark It is possible to note a posteriori that the remainder term shown here is sharp, and it gives rise to the most significant remainder appearing in Theorem 2 and Theorem 6, provided $F \neq \mathbf{Q}$.

Proof Remind that all the ideals superscrited S are prime to S. Standard estimates of the sum of ideals given by [26] lead to

$$\begin{aligned}\sum_{\substack{N\mathfrak{d}^S\leqslant X\\ \mathfrak{d}\wedge R=1}}\varphi_2(\mathfrak{d}^S) &= \sum_{\substack{N\mathfrak{l}^S\leqslant X\\ \mathfrak{l}^S\wedge R=1}}\mu^2(\mathfrak{l}^S)\sum_{\substack{N\mathfrak{m}^S\leqslant X/N\mathfrak{l}\\ \mathfrak{m}^S\wedge R=1}}N\mathfrak{m}^S\\ &= \sum_{\substack{N\mathfrak{l}^S\leqslant X\\ \mathfrak{l}^S\wedge R=1}}\mu^2(\mathfrak{l}^S)\left[\frac{\zeta^{S,R\star}(1)}{2}\frac{X^2}{(N\mathfrak{l}^S)^2}+O\left(\left(\frac{X}{N\mathfrak{l}^S}\right)^{2-\delta_F}\right)\right]\\ &= \frac{1}{2}\zeta^{S,R\star}(1)X^2\sum_{\substack{N\mathfrak{l}^S\leqslant X\\ \mathfrak{d}\wedge R=1}}\frac{\mu^2(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^2}+O\left(X^{2-\delta_F}\sum_{\substack{N\mathfrak{l}^S\leqslant X\\ \mathfrak{l}^S\wedge R=1}}\frac{\mu^2(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^{2-\delta_F}}\right)\\ &= \frac{1}{2}\frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)}X^2+\begin{cases}O(X\log X) & \text{if } F=\mathbf{Q};\\ O(X^{2-\delta_F}) & otherwise;\end{cases}\end{aligned}$$

where the knowledge of the Dirichlet series associated to μ^2 yielded

$$\sum_{N(\mathfrak{m})\leqslant X}\frac{\mu^2(\mathfrak{m})}{N\mathfrak{m}}\sim\frac{\zeta^\star(1)}{\zeta(2)}\log X=O(\log X), \tag{75}$$

in the case $F=\mathbf{Q}$, giving the worst remainder term. Otherwise, the sum is convergent. □

This lemma induces a splitting of $\nu_{1,Q}$ as $\nu_{1,Q}^{(p)}+\nu_{1,Q}^{(e)}$ according to the principal and error parts in the lemma above. □

4.3 Estimating the main part $\nu_{1,Q}^{(p)}$

Proposition 5 *For every $Q\geqslant 1$, the main part admits the asymptotic development*

$$\mathrm{vol}\,(G\,(F)\,\backslash G\,(\mathbf{A}))\,\nu_{1,Q}^{(p)}\left(\widehat{\phi}\right)=\nu\left(\widehat{\phi}\right)+O(Q^{-2}). \tag{76}$$

Proof Recall the term $\nu_{1,Q}^{(p)}$ of Proposition 4, namely

$$\begin{aligned}&\nu_{1,Q}^{(p)}(\phi)\\ &=\frac{1}{2}\frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)}\int_{\pi_S^R\in\widehat{G}_S^R}\frac{\widehat{\phi}(\pi_S^R)}{c(\pi_S^R)^2}\sum_{\substack{N\mathfrak{m}^S\leqslant Q/c(\pi_S^R)\\ \mathfrak{m}^S\wedge R=1}}\frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^2}\\ &\times\sum_{\substack{\pi_R\in\widehat{G}_R\\ c\,(\pi_R)\leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}}\frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2}\mu_R^{\mathrm{Pl}}(\pi_R)\mathrm{d}\pi_S^R.\end{aligned}$$

The following lemma states the convergence of the sum over ramified parts. □

Lemma 5 *For every* $\mathrm{Re}(s) > 1$, *the following sum converges as* $Q \to \infty$:

$$\sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^s}. \tag{77}$$

Proof The Jacquet–Langlands correspondence states a bijection between $\widehat{G}_R$ and the discrete part of the spectrum of $\widehat{\mathrm{PGL}}(2, F_R)$ by Theorem 4, which preserves both formal degrees, which are the Plancherel measures $\mu_R^{\mathrm{Pl}}(\pi_R)$, and conductors. Hence,

$$\sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\sigma_R)}{c(\sigma_R)^s} \leqslant \sum_{\pi_R \in \widehat{\mathrm{PGL}(2, F_R)}^{\mathrm{disc}}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^s}, \tag{78}$$

and that last sum is finite for $\mathrm{Re}(s) > 1$ by the case of $\mathrm{PGL}(2)$ by the computations of [9] or by Sect. 4.4 below. Hence, it follows the sought convergence for the ramified parts, ending the proof of the lemma. □

The prime-to-S-and-R part of the Dirichlet series associated to λ_2 converges at 2 to $\zeta_F^{S,R}(2)^{-2}$ and makes the expression of $\nu_{1,Q}^{(p)}$ converge to

$$\frac{1}{2} \frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)} \int_{\widehat{G}_R} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2} \mathrm{d}\pi_R \int_{\widehat{G}_S^R} \frac{\widehat{\phi}(\pi_S^R)}{c(\pi_S^R)^2} \mathrm{d}\pi_S^R. \tag{79}$$

4.4 Rewriting the constant

Previous computations unveiled the constant

$$\frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)}. \tag{80}$$

It is possible to give to this constant a more geometric flavour by reformulating the special values of the zeta functions appearing in terms of volumes. This is the content of the following lemma.

Proposition 6 *For every finite set of places S,*

$$\frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)} = \int_{\widehat{G}^{S,R}}^{\star} \frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2} = \zeta^{S,R\star}(1) \prod_{\mathfrak{p} \notin S \cup R} \zeta_{\mathfrak{p}}(1)^{-1}. \tag{81}$$

Proof The knowledge of the volumes of congruence subgroups (68) gives

$$\varepsilon_{\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)}(1) = \mathrm{vol}\left(\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)\right)^{-1} = (\mathrm{id}\star\mu^2)(\mathfrak{p}^r). \tag{82}$$

On the other hand, this volume can be computed by the Plancherel formula. Introduce the volume of slices of the spectrum of fixed conductor

$$M_{\mathfrak{p}}(\mathfrak{p}^r) = \int_{\substack{\sigma_{\mathfrak{p}} \in \widehat{G}_{\mathfrak{p}} \\ \mathfrak{c}(\sigma_{\mathfrak{p}}) = \mathfrak{p}^r}} \mathrm{d}\sigma_{\mathfrak{p}}, \qquad r \geqslant 1.$$

The Plancherel inversion formula then yields

$$\begin{aligned}\varepsilon_{\overline{K_{0,\mathfrak{p}}(\mathfrak{p}^r)}}(1) &= \int_{\widehat{G}_\mathfrak{p}} \widehat{\varepsilon_{\overline{K_{0,\mathfrak{p}}(\mathfrak{p}^r)}}}(\pi_\mathfrak{p})\mathrm{d}\pi_\mathfrak{p} = \int_{\widehat{G}_\mathfrak{p}} \tau_2\left(\frac{\mathfrak{p}^r}{\mathfrak{c}(\pi_\mathfrak{p})}\right)\mathrm{d}\pi_\mathfrak{p} \\ &= \sum_{\mathfrak{d} \mid \mathfrak{p}^r} M_\mathfrak{p}(\mathfrak{d})\tau_2\left(\frac{\mathfrak{p}^r}{\mathfrak{d}}\right) = (M_\mathfrak{p}\star\tau_2)(\mathfrak{p}^r).\end{aligned}$$

Hence, by inversion, $M_\mathfrak{p} = \mathrm{id}\star\mu^2\star\lambda_2$. In particular, the local Dirichlet series associated to $M_\mathfrak{p}$ is given by

$$D_\mathfrak{p}(s) = \sum_{\substack{\mathfrak{m}=\mathfrak{p}^r \\ r\geqslant 0}} \frac{M_\mathfrak{p}(\mathfrak{m})}{N\mathfrak{m}^s} = \frac{\zeta_\mathfrak{p}(s-1)}{\zeta_\mathfrak{p}(s)\zeta_\mathfrak{p}(2s)}, \qquad \mathrm{Re}(s) > 1. \tag{83}$$

Evaluating it at $s = 2$, a new expression for the local special values appearing in the constant is

$$\int_{\widehat{G}_\mathfrak{p}} \frac{\mathrm{d}\pi_\mathfrak{p}}{c(\pi_\mathfrak{p})^2} = \frac{\zeta_\mathfrak{p}(1)}{\zeta_\mathfrak{p}(2)\zeta_\mathfrak{p}(4)}, \tag{84}$$

proving the finiteness of the local integrals defining the equidistribution measure (6) at the finite places, as claimed in the introduction. However, the infinite product over $\mathfrak{p} \notin R$ of these quantities unfortunately diverges, for 1 is a pole of $\zeta^{S,R}$. This motivates a slight modification in order to compensate it by the residue at 1. Introduce the regularized integral

$$\int_{\widehat{G}^{S,R}}^\star \frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2} = \zeta^{S,R\star}(1)\prod_{\mathfrak{p}\notin S\cup R}\zeta_\mathfrak{p}(1)^{-1}\int_{\widehat{G}_\mathfrak{p}}\frac{\mathrm{d}\pi_\mathfrak{p}}{c(\pi_\mathfrak{p})^2} = \zeta^{S,R\star}(1)\prod_{\mathfrak{p}\notin S\cup R}\frac{1}{\zeta_\mathfrak{p}(2)\zeta_\mathfrak{p}(4)},$$

ending the proof. □

The global integral is defined to be

$$\int_{\widehat{\Pi}}^\star \frac{\mathrm{d}\pi}{c(\pi)^2} = \int_{\widehat{G}^{S,R}}^\star \frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2}\int_{\widehat{G}_{S\cup R}}\frac{\widehat{\phi}(\pi_{S\cup R})}{c(\pi_{S\cup R})^2}\mathrm{d}\pi_{S\cup R} = \zeta^\star(1)\prod_v \zeta_v(1)^{-1}\int_{\widehat{G}_v}\frac{\mathrm{d}\pi_v}{c(\pi_v)^2}.$$

It thus follows the expression (6) of the regularized integral, giving the desired statement and motivating the choice of both the measure μ and the constant C. Since the error terms appearing in the above paragraph are those of Dirichlet series at a point distant from their abscissa of convergence by 1, the expression (79) rewrites

$$\nu_{1,Q}^{(p)}(\widehat{\phi}) = \frac{1}{2}\int_{\widehat{\Pi}}^\star \frac{\widehat{\phi}(\pi)}{c(\pi)^2}\mathrm{d}\pi + O(Q^{-1}), \tag{85}$$

reaching the term of the proof of Proposition 5. □

Remark Notice that the Sauvageot theorem is a two-edged result: it opens the path to equidistribution and allows conclusions for characteristic functions which are not of the form $\widehat{\phi}$; however it also spoils the remainder term for general functions. This error term remains only for specific functions either admissible, *i.e.* of the form $\widehat{\phi}$ for ϕ in the Hecke algebra of G, or in particular for the counting problem.

4.5 Estimating the error term $\nu_{1,Q}^{(e)}$

Lemma 6 *For every $Q \geqslant 1$,*

$$\nu_{1,Q}^{(e)}(\phi) \ll Q^{-\delta_F+\varepsilon_F}, \tag{86}$$

for all $\varepsilon_F > 0$ for the case $F = \mathbf{Q}$, and for $\varepsilon_F = 0$ otherwise.

Proof Now turn back to the treatment of the error term $\nu_{1,Q}^{(e)}$ coming from the error term in Lemma 4. The bound that has to be refined is

$$\nu_{1,Q}^{(e)}(\phi) \ll Q^{-\delta_F+\varepsilon_F} \int_{\pi_S^R \in \widehat{G}_S^R} \frac{|\widehat{\phi}|(\pi_S^R)}{c(\pi_S^R)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ m^S \wedge R=1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{|\widehat{\phi}|(\pi_R)}{c(\pi_R)^{2-\delta_F+\varepsilon_F}} \mu_R^{\mathrm{Pl}}(\pi_R)\mathrm{d}\pi_S^R$$

The inner sums and integrals converge by Lemma 5, since $2 - \delta_F + \varepsilon_F$ is always greater than 1. It follows a remainder term in $Q^{-\delta_F+\varepsilon_F}$. □

At last, the asymptotic development obtained in Proposition 5 and the bounds obtained in Lemma 6 prove the equidistribution of the identity part of the counting measure with respect to ν, as stated in Proposition 3.

5 Spectral error terms

5.1 Characters contribution

Recall that the global characters contribution is given by

$$\nu_{\Xi,Q}(\widehat{\phi}) = \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R=1}} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\phi, \pi_R). \tag{87}$$

Lemma 7 *For every $\varepsilon > 0$,*

$$\nu_{\Xi,Q}(\widehat{\phi}) \ll Q^{-1+\varepsilon}. \tag{88}$$

Proof Similarly to the intervention of the trace formula to make explicit the measure ν_Q, the Poisson summation formula is the main tool to count characters. The counting measure for characters can be interpreted as a spectral side, such that every non-identity terms vanishes on the geometric side. Recall that for a character π_R, since the multiplicities are all equal to one,

$$\Xi(\pi_R, \phi) = \sum_{\substack{\chi \in X^{\mathrm{ur}}(G(\mathbf{A})) \\ \chi_R \simeq \pi_R}} \widehat{\phi}(\chi). \tag{89}$$

Consider GL(2) instead of PGL(2) for simplicity, characters of PGL(2) corresponding to those of GL(2) trivial on the center. Characters on GL(2) decompose through

$$G(F_\mathfrak{p}) \longrightarrow F_\mathfrak{p}^\times \longrightarrow S^1, \tag{90}$$

where the first arrow is given by the determinant and the second by characters of $F_\mathfrak{p}^\times$. In other words, a character $\chi_\mathfrak{p}$ of $\mathrm{GL}(2, F_\mathfrak{p})$ is of the form $\chi_{0,\mathfrak{p}} \circ \det$ where $\chi_{0,\mathfrak{p}}$ is a character of $F_\mathfrak{p}^\times$.

At an archimedean place v, since the considered characters are trivial on the center, they are among the trivial one and the sign, hence have conductor 1 at those places. Archimedean characters are of the form $\mathrm{sgn}^{\varepsilon}|\det|^{it}$ for $\varepsilon \in \{0, 1\}$ and $t \in \mathbf{R}$. It is not possible to select precisely continuous parameters, it is hence necessary to supply an approximation by a localizing function. This motivates the introduction of a compactly supported non-negative smooth function f_v such that $\widehat{f_v}$ is 1 for $t = 0$, and $|\widehat{f_v}| \leqslant 1$. In particular, it vanishes unless t is small enough, say $|t| \leqslant T$.

For the arithmetic part of the conductor, the only characters not killed by the action of $\widehat{\varepsilon}_{\mathfrak{p}^r}$ are the unramified ones. Indeed, recall that

$$\det\left(K_0(\mathfrak{p}^r)\right) = \mathcal{O}_{\mathfrak{p}}^{\times}, \tag{91}$$

so that $\chi_{0,\mathfrak{p}}$ needs to be trivial on $\mathcal{O}_{\mathfrak{p}}^{\times}$, that is to say be unramified. Introduce, for every finite split place $\mathfrak{p}$, the characteristic function $f_{\mathfrak{p}}$ of $\mathcal{O}_{\mathfrak{p}}^{\times}$, whose Fourier transform selects unramified characters analogously to Lemma 3. Introduce the global test function

$$f = \prod_{\mathfrak{p}\notin R} f_{\mathfrak{p}} \prod_{v\in R} \xi_{\chi_v} \prod_{\substack{v\mid\infty \\ v\notin R}} f_v. \tag{92}$$

Since $\hat{f}_{\mathfrak{p}}$ is 1 on unramified characters and the archimedean $\hat{f}_v$'s are less than one, the Poisson summation formula gives

$$\Xi(\pi_R, \phi) \ll \sum_{\chi\in\widehat{F^{\times}}} \widehat{f}(\chi) = \frac{1}{\mathrm{vol}(F^{\times}\backslash\mathbf{A}^{\times})} \sum_{\gamma\in F^{\times}} f(\gamma). \tag{93}$$

Since $F^{\times}$ is a discrete set, choosing f_{∞} with a small enough support leads to kill every $f(\gamma)$ for γ nontrivial. Hence $\Xi(\pi_R, \phi) \leqslant \mathrm{vol}(F^{\times}\backslash\mathbf{A}^{\times})^{-1} f(1)$. It remains to evaluate $f(1) = f_{\mathfrak{d}^S}(1) f_R(1) f_{\infty}(1)$. For the finite split places, $f_{\mathfrak{p}}(1) = 1$, and for the ramified places, $f_R(1) = \mu_R^{\mathrm{Pl}}(\pi_R)$. For the archimedean places, the Plancherel inversion formula gives

$$f_{\infty}(1) = \int_{\widehat{F}_{\infty}^{R}} \widehat{f_{\infty}}(\chi)\mathrm{d}\chi \leqslant \int_{|t|\leqslant T} \mathrm{d}\chi_t \ll_T 1. \tag{94}$$

Finally, $\Xi(\pi_R, \phi) \ll \mu_R^{\mathrm{Pl}}(\pi_R)$. Coming back to the sum (87) defining $\nu_{\Xi,Q}(\widehat{\phi})$, it follows by using the rough bound $\lambda_2(\mathfrak{n}) \ll N\mathfrak{n}^{\varepsilon}$,

$$\begin{aligned}
\nu_{\Xi,Q}(\widehat{\phi}) &\ll \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \mu_R^{\mathrm{Pl}}(\pi_R) \\
&\ll \frac{1}{Q^2} \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q}} \mu_R^{\mathrm{Pl}}(\pi_R) \sum_{\substack{N\mathfrak{d}\leqslant Q/c(\pi_R) \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{N\mathfrak{m}\leqslant Q/N\mathfrak{d}c(\pi_R) \\ \mathfrak{q}\wedge R=1}} N\mathfrak{m}^{\varepsilon} \\
&\ll Q^{-1+\varepsilon} \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^{1+\varepsilon}} \sum_{\substack{N\mathfrak{d}\leqslant Q/c(\pi_R) \\ \mathfrak{q}\wedge R=1}} N\mathfrak{d}^{-1-\varepsilon} \\
&\ll Q^{-1+\varepsilon}
\end{aligned}$$

and this last line is provided by the convergence of the sum over ramified representation, stated in Lemma 5, proving the result. □

6 Elliptic error terms

The present section aims at bounding the different terms appearing in the elliptic contribution to the geometric side, in particular the orbital integrals. A considerable amount of work has been done in this direction, and we borrow recent results of Binder [6] in order to reach our goal.

6.1 Strategy

The contribution of the elliptic terms in the trace formula (46) is

$$J_{\text{ell}}(\Phi) = \sum_{\{\gamma\}\neq 1} \text{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x. \tag{95}$$

Recall that Φ denotes the test function $\Phi_{\mathfrak{d},\pi_R;\phi}$ introduced in Sect. 3.4, and that only a subset of the indices $(\mathfrak{d}, \pi_R; \phi)$ may be used when the dependency on them has to be emphasized. As a matter of fact, the expression (95) generally requires to bound

- the length of the summation, provided it is finite;
- the global volumes vol $\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right)$;
- the orbital integrals.

Since for every finite place $\mathfrak{p}$, the test function $\Phi_\mathfrak{p}$ is supported on either $K_\mathfrak{p}$ in the case of a split place, or on $G_\mathfrak{p}$ in the case of a ramified place, it is compactly supported on a compact independent of the chosen spectral parameters. Since $G(F)$ is discrete in $G(\mathbf{A})$, there is only a finite number of classes contributing to (95), and this number is uniformly bounded with respect to the choices of spectral parameters. Therefore, the associated global volumes are also uniformly bounded, so that only the orbital integrals remain to be bounded.

Proposition 7 *For every $\gamma \in G(F)$, there is a $c > 0$ such that*

$$\mathcal{O}_\gamma(\Phi) \ll_\varepsilon (N\mathfrak{d}^R)^{-1+\varepsilon} \mu_R^{\text{Pl}}(\pi_R). \tag{96}$$

The next subsections are devoted to the proof of this proposition. The local components $\Phi_\mathfrak{p}$ are almost always equal to $\mathbf{1}_{\overline{K}_\mathfrak{p}}$, so that the corresponding local orbital integrals are almost always trivial by the normalizations of measures. Following [25], the local decomposition of orbital integrals for factorizable functions $\Phi = \otimes_v \Phi_v$ then holds, more precisely

$$\mathcal{O}_\gamma(\Phi) = \prod_v \mathcal{O}_{\gamma,v}(\Phi_v) \quad \text{where} \quad \mathcal{O}_{\gamma,v}(\Phi_v) = \int_{G_{\gamma,v}\backslash G_v} \Phi_v\left(x_v\gamma_v x_v^{-1}\right) \mathrm{d}x_v. \tag{97}$$

It thus suffices to dominate these local orbital integrals. The split and ramified cases behave quite differently and require specific treatments.

6.2 Split orbital integrals

Almost every place is split, thus precise bounds are needed in order to control the global orbital integral. Fortunately, the test functions chosen at these places are explicit and allows to sharply control the associated orbital integrals.

6.2.1 Non-archimedean split places

Lemma 8 *For every $\gamma^R \in G^R$, every ideal $\mathfrak{d}^R$ of $\mathcal{O}^R$ and every $\varepsilon > 0$,*

$$\mathcal{O}_{\gamma^R}(\mathfrak{d}^R) \ll \prod_{\mathfrak{p}^r || \mathfrak{d}^R} |D(\gamma)|_{\mathfrak{p}}^{-1} N\left(\mathfrak{p}^r\right)^{\varepsilon}. \tag{98}$$

Proof By the local factorization of orbital integral, it is sufficient to prove the lemma for a fixed place. Let $\mathfrak{p} \notin R$ and $\gamma_{\mathfrak{p}} \in G_{\mathfrak{p}}$. In the case of a place $\mathfrak{p} \notin S$, the local test function is of the form $\varepsilon_{\mathfrak{p}^r}$, so that

$$\mathcal{O}_\gamma(\varepsilon_{\mathfrak{p}^r}) = \mathrm{vol}(K)^{-1}\mathcal{O}_\gamma(\mathbf{1}_K), \qquad \text{where} \qquad K = \overline{K}_0(\mathfrak{p}^r). \tag{99}$$

Bounds for the split orbital integrals are provided by Binder [6, Proposition 8.2.1] in the specific case of GL(2), and yield the following estimate depending on $\gamma_{\mathfrak{p}}$:

$$\mathcal{O}_{\gamma_{\mathfrak{p}}}\left(\varepsilon_{\mathfrak{p}^r}\right) \ll |D(\gamma)|_{\mathfrak{p}}^{-1} N(\mathfrak{p}^r)^{-1+\varepsilon}\mathrm{vol}(\overline{K}_0(\mathfrak{p}^r))^{-1} \ll |D(\gamma)|_{\mathfrak{p}}^{-1} N\left(\mathfrak{p}^r\right)^{\varepsilon}. \tag{100}$$

Otherwise, for $\mathfrak{p} \in S$, the chosen test function is $\phi_{\mathfrak{p}}$ and hence can be roughly bounded by

$$\mathcal{O}_{\gamma_{\mathfrak{p}}}(\phi_{\mathfrak{p}}) \ll |D(\gamma)|_{\mathfrak{p}}^{-1}, \tag{101}$$

settling the desired estimates for finite split orbital integrals. □

6.3 Non-split orbital integrals

Ramified places are in finite number but the explicit behavior of local orbital integrals could a priori be unbounded. Underlining that the sum over conjugacy classes appearing as the geometric side of the trace formula is uniformly bounded, we can afford a dependence on γ_v at ramified places. We have the following.

Lemma 9 *For every ramified place v,*

$$\mathcal{O}_{\gamma_v}(\Phi_v) \ll |D(\gamma)|_v^{-1/2}\mu_v^{\mathrm{Pl}}(\pi_v). \tag{102}$$

Proof Archimedean and non-archimedean ramified places behave differently. Before turning to the precise study of each case, note that whatever Φ_v is ξ_{π_v} or $\xi_{\pi_v}\widehat{\phi_v}(\pi_v)$, the orbital integral is dominated by the case of the matrix coefficient ξ_{π_v}, for $\widehat{\phi_v}$ is bounded. In the ramified case, orbital integrals are characters: for a representation $\pi_v \in \widehat{G}_v$, the main geometric lemma of Arthur [2] implies that

$$\mathcal{O}_{\gamma_v}(\xi_{\pi_v}) \ll \Theta_{\pi_v}(\gamma_v)\mu_v^{\mathrm{Pl}}(\pi_v), \tag{103}$$

where Θ_{π_v} stands for the Harish–Chandra character associated to π_v. It is in particular sufficient to bound characters on $B_{\mathfrak{p}}^{\times}$ in order to get the desired bound for orbital integrals.

6.3.1 Archimedean ramified places

For matrix coefficients, we follow the work of Kim, Shin and Templier [21, Proposition 5.1]. Since all the elements γ_v are elliptic and regular, the Harish–Chandra character formula implies that

$$\Theta_{\pi_v}(\gamma_v) \ll |D(\gamma)|_v^{-1/2}. \tag{104}$$

6.3.2 Non-archimedean ramified places

Concerning the non-archimedean ramified places $\mathfrak{p} \in R$, the lead is given to Shin and Templier [41], who build on the Sally-Shalika character formula in order to give explicit computations for the characters of each supercuspidal representations of SL(2). They prove that for every supercuspidal representation $\pi_\mathfrak{p}$ of $\mathrm{SL}(2, F_\mathfrak{p})$, and for all semisimple regular element $\gamma_\mathfrak{p}$,

$$\Theta_{\pi_\mathfrak{p}}(\gamma_\mathfrak{p}) \ll |D(\gamma)|_\mathfrak{p}^{-1/2}. \tag{105}$$

Moreover, it suffices to achieve this goal for SL(2). Indeed, Langlands and Labesse [28] established that every irreducible admissible representation of GL(2) restricts to a direct sum of at most four irreducible admissible representations of SL(2). Since the Jacquet–Langlands correspondence maps irreducible representations of $G_\mathfrak{p}$ to supercuspidal representations by Theorem 4, and the image of the embedding of $G_\mathfrak{p}$ in $\mathrm{GL}(2, F_\mathfrak{p})$ is made of semisimple regular elements, the bound above apply to $B_\mathfrak{p}$, and therefore to $G_\mathfrak{p}$.

The bounds obtained in the two cases of ramified places hence settle the proof of Lemma 9. Moreover, since there are only a uniformly finite number of conjugacy classes γ contributing in the geometric side of the trace formula, the Weyl's discriminants $D(\gamma)$ are uniformly bounded, achieving the proof of Proposition 7. □

6.4 Final estimates

All the tools are now in place to establish the final estimates on the elliptic contribution $\nu_{\mathrm{ell},Q}(\widehat{\phi})$ and reach the term of the proof of Theorem 6.

Proposition 8 *Let $d = [F : \mathbf{Q}]$. For a finite set of places S, $\phi \in \mathcal{H}(G_S)$ and any $\varepsilon > 0$, the elliptic contribution is dominated by*

$$\nu_{\mathrm{ell},Q}\left(\widehat{\phi}\right) \ll Q^{-1/d+\varepsilon}. \tag{106}$$

Proof The bounds stated in Proposition 7 and the definition of the elliptic contribution lead to

$$\begin{aligned} J_{\mathrm{ell}}\left(\Phi_{\mathfrak{d},\pi_R;\phi}\right) &= \sum_{\{\gamma\}\neq 1} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma\left(\Phi_{\mathfrak{d},\pi_R;\phi}\right) \\ &\ll (N\mathfrak{d}^S)^\varepsilon \mu_R^{\mathrm{Pl}}(\pi_R). \end{aligned}$$

Summing over the spectral data leads to

$$\begin{aligned} \nu_{\mathrm{ell},Q}\left(\widehat{\phi}\right) &\ll \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)(N\mathfrak{d}^S)^\varepsilon \\ &\quad \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S}} \widehat{\phi}(\pi_R)\mu_R^{\mathrm{Pl}}(\pi_R) \\ &\ll Q^{-1/d} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)(N\mathfrak{d}^S)^{\varepsilon+1/d-2} \end{aligned}$$

$$\sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}^S}} \frac{\widehat{\phi}(\pi_R)\mu_R^{\text{Pl}}(\pi_R)}{c(\pi_R)^{2-1/d}}$$

where the elementary bound $\lambda_2(\mathfrak{n}) \ll_\varepsilon N\mathfrak{n}^\varepsilon$ has been used. Thus, since Lemma 5 ensures the convergences of the inner sum, it follows that

$$\nu_{\text{ell},Q}\left(\widehat{\phi}\right) \ll_\varepsilon Q^{-1/d+\varepsilon}$$

This achieves the proof that the main term contributing in (64) is the one coming from the identity as stated in Proposition 3, hence also Theorems 2, 3 and 6. □

7 Sato-Tate corollary

Theorem 3 proves the existence of a measure ν with respect to which the universal family equidistributes. Consider the projection $\nu_\mathfrak{p}$ of ν on the local components $\widehat{G}_\mathfrak{p}$. Since the $\nu_\mathfrak{p}$ are supported on different spaces, it is necessary to make sense of the Sato-Tate problem that concerns convergence of the measures $\nu_\mathfrak{p}$.

The literature often treats the case of measures supported on the unramified tempered spectrum, as the instances handled by Sarnak [36] or Serre [40]. In those cases, the Satake isomorphism provides a common parametrization: if T is the standard torus of $\text{SL}(2, \mathbf{C})$, the dual group of $\text{PGL}(2)$, and W is the associated Weyl group, then the isomorphism classes of unramified tempered representations are parametrized by T_c/W where $T_c = T \cap SU(2, \mathbf{C})$ is the compact part of T. This last quotient corresponds to the half-circle, giving a common ground for all the $\widehat{G}_\mathfrak{p}$,independent of $\mathfrak{p}$. Even if the universal family considered does include ramified representations and the $\nu_\mathfrak{p}$ are supported on the whole tempered unitary dual, the contribution of the ramified part of the spectrum vanish when $\mathfrak{p}$ goes to infinity, so that asymptotically the spaces can be identified and T_c/W is a posteriori still a relevant common ground to state the Sato-Tate result.

For $\text{GL}(2, F_\mathfrak{p})$, the Plancherel measures have been computed by Serre [40] and are given by

$$\text{d}\mu_\mathfrak{p}^{\text{Pl}}(x) = \frac{N\mathfrak{p}+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(N\mathfrak{p}^{1/2}+N\mathfrak{p}^{-1/2})^2 - x^2}\text{d}x. \tag{107}$$

In particular they converge, as $N\mathfrak{p}$ goes to infinity, to the Sato-Tate measure on the half-circle

$$\text{d}\mu^{\text{ST}}(x) = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\text{d}x, \tag{108}$$

in the sense that for any $\widehat{\phi} \in C(T_c/W, \mathbf{C})$, when $N\mathfrak{p}$ goes to infinity,

$$\int_{T_c/W} \widehat{\phi}(\pi_\mathfrak{p})\text{d}\mu_\mathfrak{p}^{\text{Pl}}(\pi_\mathfrak{p}) \longrightarrow \int_{T_c/W} \widehat{\phi}(x)\text{d}\mu^{\text{ST}}(x). \tag{109}$$

For $\widehat{\phi} \in C(T/W, \mathbf{C})$, let decompose the measure separating whether the representations are unramified, *i.e.* of conductor 1, or not. The measure $\nu_{\mathfrak{p}}(\widehat{\phi})$ hence splits as

$$\begin{aligned}\int_{\widehat{G}_{\mathfrak{p}}} \widehat{\phi}(\pi_{\mathfrak{p}}) \mathrm{d}\nu_{\mathfrak{p}}(\pi_{\mathfrak{p}}) &= \int_{\widehat{G}_{\mathfrak{p}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) \\ &= \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}} \widehat{\phi}(\pi_{\mathfrak{p}}) \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) + \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}),\end{aligned} \tag{110}$$

where $\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}$ stands for the unramified, also called spherical, part of the spectrum and $\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}$ for its ramified part. For $\mathfrak{p}$ sufficiently large, $G_{\mathfrak{p}}$ is isomorphic to $\mathrm{PGL}(2, F_{\mathfrak{p}})$, so the local Plancherel measures (107) provide the value of the first integral of the rightmost hand side as $\mathfrak{p}$ grows, in particular they converge to the Sato-Tate measure. For the second one, dominating roughly by leaving the dependence in ϕ which is fixed gives

$$\int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) \ll \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} = \int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} - \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}} \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}). \tag{111}$$

By the normalization of the Plancherel measure, the second integral on the right hand side is 1. Moreover, as shown in Sect. 4.4, the first integral of the right hand side is equal to

$$\int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} = \frac{\zeta_{\mathfrak{p}}(1)}{\zeta_{\mathfrak{p}}(2)\zeta_{\mathfrak{p}}(4)}. \tag{112}$$

Since this last quantity is $1 + O(N\mathfrak{p}^{-1})$ by unfolding the definition of the Dirichlet series, it follows that the ramified part is negligible, achieving the proof of Corollary 1. □

Acknowledgements I am infinitely indebted to my advisor, Farrell Brumley, for having trusted in me for handling this problem and for having been strongly present and implied in its resolution. I am grateful to the time Gergely Harcos and Philippe Michel granted me in carefully reading the thesis from which this article blossomed. I would like to thank Valentin Blomer, Ernest Hunter Brooks, Andrew Corbett, Mikolaj Fraczyk, Élie Goudout, Guy Henniart and Ian Petrow for many enlightening discussions. This work has been faithfully supported by the ANR 14-CE25 PerCoLaTor, the Fondation Sciences Mathématiques de Paris and the Deutscher Akademischer Austauschdienst. At last, nothing would have been done without the warm and peaceful environments of the institutions which hosted me during these years: Université Paris 13, École Polytechnique Fédérale de Lausanne, Georg-August Universität and Sun Yat-Sen University.

References

1. Adler, J.D., DeBacker, S., Sally, P.Jr., Spice, L.: Supercuspidal characters of SL(2) over a p-adic field. In: Harmonic Analysis on Reductive, p-adic Groups. Amer. Math. Soc. pp. 19–69 (2011)
2. Arthur, J.: The characters of supercuspidal representations as weighted orbital integrals. Proc. Indian Acad. Sci. Math. Sci. **97**, 3–19 (1988)
3. Arthur, J.: An introduction to the trace formula. In: Harmonic analysis, the trace formula, and Shimura varieties. Clay. Math. Proc. vol. 4, pp. 1–263 (2005)
4. Batyrev, V.V., Manin, Y.I.: Sur le nombre des points rationnels de hauteur bornée des variétés algébriques. Math. Ann. **286**, 27–43 (1990)
5. Bernstein, J., Deligne, P., Kazhdan, D.: Trace paley-wiener theorem for reductivep-adic groups. J. Anal. Math. **47**, 180–192 (1986)
6. Binder, J.: Fields of Rationality of Cusp Forms. Israel J. Math. **222**, 973–1028 (2017)
7. Brooks, E.H., Petrow, I.: Counting automorphic forms on norm one tori. Acta Arith. **183**, 117–143 (2018)
8. Brumley, F.: Effective Multiplicity One for GL(N) and narrow zero-free regions for Rankin–Selberg L-functions. Am. J. Math. **128**, 1455–1474 (2006)
9. Brumley, F., Milićević, D.: Counting cusp forms with analytic conductor (2018). Preprint arXiv:1805.00633

10. Casselman, W.: On Some Results by Atkin and Lehner. Math. Ann. **201**, 301–314 (1973)
11. Chambert-Loir, A., Tschinkel, Y.: Igusa integrals and volume asymptotics in analytic and adelic geometry. Confluentes Math. **2**, 351–429 (2010)
12. Clozel, L., Delorme, P.: Le théorème de Paley-Wiener invariant pour les groupes réductifs. II. Ann. Sci. ÉNS. **23**, 193–228 (1990)
13. Conrey, J.B., Farmer, D.W., Keating, J.P., Rubinstein, M.O., Snaith, N.C.: Integral moments of L-functions. Proc. Lond. Math. Soc. **91**, 33–104 (2005)
14. Diamond, F., Shurman, J.: A First Course in Modular Forms. Graduate Texts in Math 228. Springer (2005)
15. Gelbart, S.: Automorphic Forms on Adele Groups. Ann. Math. Studies. 83, Princeton University Press, Princeton (1975)
16. Godement, R., Jacquet, H.: Zeta Functions of Simple Algebras. Lecture Notes in Math 260. Springer, Berlin (1972)
17. Hare, K.E.: The size of characters of compact Lie groups. Stud. Math. **129**, 1–18 (1998)
18. Henniart, G.: On the local Langlands and Jacquet–Langlands correpsondence. International Congress of Mathematicians. II, pp. 1171–1182 (2006)
19. Hull, R.: The maximal order of generalized quaternion division algebras. Bull. Am. Math. Soc. **22**, 1–11 (1936)
20. Iwaniec, H., Sarnak, P.: Perspectives on the Analytic Theory of L-functions. Geom. Funct. Anal. pp. 705–741 (2000)
21. Kim, J.-L., Shin, S.W., Templier, N.: Asymptotics and Local Constancy of Characters of p-Adic Groups. In: Simons Symposia. Springer, pp. 259–295 (2016)
22. Knapp, A.W.: Representation Theory of Semisimple Groups: An Overview based on Examples. Princeton Landmarks in Math 36 Princeton University Press, Princeton (1986)
23. Knapp, A.W.: Local Langlands Correspondence The Archimedean Case. In: Proc. Symp. Pure Math 55, 393–410. Amer. Math. Soc. (1994)
24. Knapp, A.W., Trappa, P.E.: Representation of semisimple lie groups. In: Representation Theory of Lie Groups. Amer. Math. Soc, pp. 5–88 (2000)
25. Knightly, A., Li, C.: Traces of Hecke operators. Math. Surveys and Monographs 133. Amer. Math. Soc (2006)
26. Landau, E.: Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. B. G. Teubner, Leipzig (1918)
27. Lang, S.: SL(2, R), Graduate Texts in Mathematics 105. Springer, Berlin (1985)
28. Langlands, R.P., Labesse, J.-P.: L-indistinguishability for SL(2). Can. J. Math. **31**, 726–785 (1979)
29. Lansky, J., Raghuram, A.: On the correspondence of representations between GL(n) and division algebras. Proc. Am. Math. Soc. **131**, 1641–1648 (2004)
30. Lesesvre, D.: Arithmetic Statistics for Quaternion Algebras. PhD dissertation, Université Paris 13 (2018)
31. Northcott, D.G.: Periodic points on an algebraic variety. Ann. Math. **51**, 167–177 (1950)
32. Petrow, I.: The Weyl law on algebraic tori (2018). Preprint https://arxiv.org/pdf/1808.09991.pdf
33. Peyre, E.: Hauteurs et mesures de Tamagawa sur les variétés de Fano. Duke J. Math. **79**, 101–218 (1995)
34. Prasad, D., Raghuram, A.: Representation Theory of GL(n) over Non-Archimedean Local Fields. Lectures given at the School on Automorphic Forms on GL(n). pp. 1-48 (2000)
35. Sally, P.J., Shalika, J.: Characters of the discrete series of representations of SL(2) over a local field. Proc. Natl. Acad. Sci. USA **61**, 1231–1237 (1968)
36. Sarnak, P.: Statistical properties of eigenvalues of the hecke operators. In: Analytic Number Theory and Diophantine Problems. Birkhäuser, pp. 321–331 (1987)
37. Sarnak, P., Shin, S.W., Templier, N.: Families of L-functions and their symmetries. In: Families of automorphic forms and the trace formula. Springer, pp. 531–578 (2016)
38. Sauvageot, F.: Principe de densité pour les groupes réductifs. Compos. Math. **108**, 151–184 (1997)
39. Schanuel, S.: On heights in number fields. Bull. Am. Math. Soc. **70**, 262–263 (1964)
40. Serre, J.-P.: Répartition asymptotique des valeurs propres de l'opérateur de Hecke T(p). J. Am. Math. Soc. **10**, 75–102 (1997)
41. Shin, S.W., Templier, N.: Sato-Tate theorem for families and low-lying zeros of automorphic L-functions. Invent. Math. **1**, 1–177 (2016)
42. Waldspurger, J.-L.: La formule de Plancherel pour les groupes p-adiques, d'après Harish-Chandra. J. Inst. Math Jussieu. **2**, 235–333 (2003)