Adèle rings and Tate's thesis

A (tentative of) friendly introduction

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What is important? Understanding the *adèles* $\mathbb{A}$

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$$= \int_1^\infty \frac{\Theta(t) - 1}{2} t^{(1-s)/2} \frac{\mathrm{d}t}{t} + \int_1^\infty \frac{t^{1/2} - 1}{2} t^{-s/2} \frac{\mathrm{d}t}{t}$$

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Thus, $\Lambda(s) = I(s) + I(1-s) - \left(\frac{1}{s} + \frac{1}{1-s}\right)$.

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Conclusion : need for a more efficient framework

The adelic setting

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- the analysis on such a space is essentially... empty

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In fact, for p-adic $|\cdot|_p$ we have the *ultrametric* inequality :

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$$(x_\infty, x_2, x_3, \ldots) + (y_\infty, y_2, y_3, \ldots) = (x_\infty + y_\infty, x_2 + y_2, \ldots)$$
$$(x_\infty, x_2, x_3, \ldots) \cdot (y_\infty, y_2, y_3, \ldots) = (x_\infty y_\infty, x_2 y_2, \ldots)$$

It is a topological ring with basis of open sets :

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The projection maps $x = (x_v)_v \mapsto x_v$ are continuous, i.e. the restricted product topology is finer than the product topology.

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Approximation theorems are the analogous to the Chinese remainder theorem in the *adelic setting*

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Further : All these are true for GL_n while S nontrivial

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The adèles are *self-dual*, and this allows nice Fourier analysis

Haar measure on G

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On $\mathbb{R}$, the characters are $\chi_y := x \mapsto e^{2i\pi xy} \in \widehat{\mathbb{R}}$ for $y \in \mathbb{R}$ and indeed we recover the classical Fourier transform :

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We obtain the archimedean factor as a zeta integral.

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Tip to compute *p*-adic integrals : slice by valuations (functions are locally constant, so integrals are sums)

$$\mathbb{Z}_p \backslash \{0\} = \bigsqcup_{n \geqslant 0} p^n \mathbb{Z}_p^\times$$

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Recall that $f_p = 1_{\mathbb{Z}_p}$ is self-dual : $\widehat{f_p} = f_p$

Introduce the p-adic zeta integral

$$\begin{aligned} Z(f_p, s) &:= \int_{\mathbb{Q}_p^\times} f_p(x_p)|x_p|_p^s \mathrm{d}^\times x_p = \int_{\mathbb{Z}_p \setminus \{0\}} |x_p|_p^s \mathrm{d}^\times x_p \\ &= \sum_{k \geqslant 0} \int_{p^k \mathbb{Z}_p^\times} |x_p|_p^s \mathrm{d}^\times x_p = \frac{p}{p-1} \sum_{k \geqslant 0} \int_{p^k \mathbb{Z}_p^\times} |x_p|_p^{s-1} \mathrm{d}x_p \\ &= \frac{p}{p-1} \sum_{k \geqslant 0} p^{-k(s-1)} \mathrm{vol}(p^k \mathbb{Z}_p^\times) \end{aligned}$$

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We obtain the p-factor as zeta integral

Tate's thesis

Adelic Mellin transform

Let $f_{\mathbb{A}} = \prod_v f_v$ the adelic self-dual function

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The adelic zeta integral directly appears as the completed ζ

Another proof of the Euler product

Recall $\mathbb{A}^\times/\mathbb{Q}_+^\times \simeq \mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times =: \mathbb{R}^\times \times \widehat{\mathbb{Z}}^\times$

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This only uses the Dirichlet series definition of ζ

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Note : the fundamental theorem of arithmetic lurks behind !

Adelic Poisson summation formula

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For $f \in \mathcal{S}(\mathbb{A})$, we have $\displaystyle\sum_{k \in \mathbb{Q}} f(k) = \sum_{k \in \mathbb{Q}} \widehat{f}(k)$

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This is **Tate's thesis**

Towards automorphic forms

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For a "nice" algebraic group G, and automorphic form is

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Question : What is the relation with classical objects ?

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Thank you !

非常感谢！