Adèle rings and Tate's thesis

A (tentative of) friendly introduction

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The aim of number theory

Aim : study $\mathbb{Q}$ and its extensions (prime numbers, diophantine equations, class field theory, Langlands program...)

Problem : we don't know how to do analysis on $\mathbb{Q}$

Idea : understand how to do so in a *simple* setting

- simplest example of group : GL_1
- simplest example of automorphic repr. : 1
- simplest example of L-functions : $\zeta(s)$

An overview of the contents

- Riemann's proof of the functional equation
- ring of adèles
- harmonic analysis on groups
- Tate's proof of the functional equation

What is important ? Understanding the *adèles* $\mathbb{A}$

The classical theory

Riemann ζ function

Riemann zeta function : $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ converges for $\Re(s) > 1$

Euler product decompositon : $\zeta(s) = \prod_p (1 - p^{-1})^{-s}$

Theorem (Riemann)

$\zeta(s)$ extends meromorphically to $\mathbb{C}$, with a simple pole at $s = 1$.
It satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s) \quad \text{where} \quad \Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It is a central result (e.g. prime number theorem) How to prove this functional equation ?

Theta function

Introduce the theta function $\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$.

Functional equation for Θ

For all $t > 0$, we have $\Theta(t) = t^{-1/2}\Theta\left(\frac{1}{t}\right)$.

Proof. The Gaussian $f(x) = e^{-\pi x^2}$ satisfies $\widehat{f} = f$.

Let $f_t(x) := f(\sqrt{t}x) = e^{-\pi x^2 t}$. Then $\widehat{f_t}(x) = t^{-1/2} f(x/\sqrt{t})$

Applying Poisson summation formula,

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = t^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / t} \quad \square$$

Let $\widetilde{\Theta}(t) = \frac{\Theta(t)-1}{2} = \sum_{n \geqslant 1} e^{-\pi n^2 t} = \sum_{n \geqslant 1} e^{-\pi n^2 t}$ (regularized Θ)

Riemann's classical proof

We now prove the functional equation

$$\pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) = \pi^{-\frac{s}{2}} n^{-s} \int_0^\infty e^{-x} x^{\frac{s}{2}} \frac{\mathrm{d}x}{x} = \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}} \frac{\mathrm{d}t}{t}$$

Summing over n (loc. unif. convergent for $\Re(s) > 1$),

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \int_0^\infty \widetilde{\Theta}(t) t^{\frac{s}{2}} \frac{\mathrm{d}t}{t}.$$

Riemann's classical proof

We split $(0,\infty)$ into $(0,1)$ and $(1,\infty)$:

- $I(s) := \int_1^\infty \tilde{\Theta}(t) t^{s/2} \frac{\mathrm{d}t}{t}$ converges and holomorphic
- Change variable $t \mapsto t^{-1}$ and use functional equation :

$$\int_0^1 \tilde{\Theta}(t) t^{s/2} \frac{\mathrm{d}t}{t} = \int_1^\infty \tilde{\Theta}(t^{-1}) t^{-s/2} \frac{\mathrm{d}t}{t} = \int_1^\infty \frac{t^{1/2}\Theta(t) - 1}{2} t^{-s/2} \frac{\mathrm{d}t}{t}$$

$$= \int_1^\infty \frac{\Theta(t) - 1}{2} t^{(1-s)/2} \frac{\mathrm{d}t}{t} + \int_1^\infty \frac{t^{1/2} - 1}{2} t^{-s/2} \frac{\mathrm{d}t}{t}$$

$$= I(s-1) - \frac{1}{1-s} - \frac{1}{s}.$$

Thus, $\Lambda(s) = I(s) + I(1-s) - \left(\frac{1}{s} + \frac{1}{1-s}\right)$.

Symmetry $s \leftrightarrow 1-s$, poles at 0 and 1. $\square$

Beyond the classical ζ

There are more general L-functions :

- $\zeta_K(s) = \sum_{\mathfrak{a} \in \mathcal{I}(\mathcal{O}_K)} N\mathfrak{a}^{-s}$ where $K/\mathbb{Q}$ finite (Dedekind)
- $L(s,\chi) = \sum_{n \geqslant 1} \frac{\chi(n)}{n^s}$ where $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ (Dirichlet)
- $L(s,E) = \sum_{n \geqslant 1} \frac{a_n(E)}{n^s}$, $a_p(E) = p + 1 - |E(\mathbb{F}_p)|$ (EC)
- $L(s,\rho)$ where $\rho : \mathrm{Gal}(K/k) \to \mathrm{GL}_n(\mathbb{C})$ (Galois repr.)

Some analogous results hold with similar (but trickier) proofs.

Some issues in this classical approach

- too sensitive to the base field (e.g. units, class group)
- completing factor $\Gamma(s)$ appears *ad hoc* (other fields ?)

Conclusion : need for a more efficient framework

The adelic setting

Absolute values on a field

An *absolute value* $|\cdot| : \mathbb{Q} \to \mathbb{R}$ is

- $|u+v| \leqslant |u| + |v|$ (triangle inequality)
- $|uv| = |u| \cdot |v|$ (multiplicative)
- $|u| = 0 \Leftrightarrow u = 0$ (separation)

It induces a *metric* $d(u,v) = |u-v|$.

Example : trivial absolute value $|x|_0 = 1_{x \neq 0}$

- $\mathbb{Q}$ is complete for the induced topology
- sequences converge only when they become constant
- subsets are open and closed
- the analysis on such a space is essentially... empty

Absolute values on a field

An absolute value $|\cdot|:\mathbb{Q}\to\mathbb{R}$ is

- $|u+v|\leqslant|u|+|v|$ (triangle inequality)
- $|uv|=|u|\cdot|v|$ (multiplicative)
- $|u|=0\Leftrightarrow u=0$ (separation)

It induces a metric $d(u,v)=|u-v|$.

Ostrowski

The only equivalence classes ($|\cdot|\sim|\cdot|^{\alpha}$) on $\mathbb{Q}$ are

- **archimedean** $|\cdot|_\infty$ from $\mathbb{Q}\hookrightarrow\mathbb{R}$
- **p-adic** e $|\cdot|_p$ defined by $|\frac{a}{b}p^n|_p=p^{-n}$, $(ab,p)=1)$

In fact, for p-adic $|\cdot|_p$ we have the *ultrametric* inequality :

$$|u+v|_p\leqslant\max(|u|_p,|v|_p)$$

Completions of $\mathbb{Q}$

Endow $\mathbb{Q}$ with these topologies, and complete it :

- $\mathbb{Q}_\infty = \mathbb{R}$ the real numbers of the form $x = \sum_{n \geqslant N} x_{-n} 10^{-n}$
- $\mathbb{Q}_p$ the p-adics numbers of the form $x = \sum_{n \geqslant N} x_n p^n$

The rings of integers are :

- $\mathbb{Z}_\infty = \mathbb{Z} \subset \mathbb{R}$ corresponding to $N = 0$
- $\mathbb{Z}_p \subset \mathbb{Q}_p$ corresponding to $N = 0$

The p-adic topology is deeply different :

$$\mathbb{Z}_p = \overline{B}(0,1) \text{ is a compact ring!}$$

Gluing completions together

We would like to glue the completions together, e.g.

$$\widehat{\mathbb{Z}} := \prod_p \mathbb{Z}_p \ (= \varprojlim \mathbb{Z}/N\mathbb{Z}) \text{ is compact}$$

But the corresponding manipulation on fields *fail* :

$$\prod_p \mathbb{Q}_p \text{ is not even locally compact!}$$

Indeed, the product topology is defined by the neighborhoods

$$\prod_{v \in S} U_v \prod_{v \notin S} \mathbb{Q}_v .$$

and they are *not* compact

The ring of adèles

The adèles are a *restricted product* :

$$\mathbb{A} = \prod_v (\mathbb{Q}_v, \mathbb{Z}_v) := \left\{ (x_v)_v \in \prod_v \mathbb{Q}_v \ : \ \text{a.e. } x_v \in \mathbb{Z}_v \right\}$$

$$(x_\infty, x_2, x_3, \ldots) + (y_\infty, y_2, y_3, \ldots) = (x_\infty + y_\infty, x_2 + y_2, \ldots)$$
$$(x_\infty, x_2, x_3, \ldots) \cdot (y_\infty, y_2, y_3, \ldots) = (x_\infty y_\infty, x_2 y_2, \ldots)$$

It is a topological ring with basis of open sets :

$$\prod_v U_v, \ \ U_v \text{ open of } \mathbb{Q}_v \text{ and a.e. } U_v = \mathbb{Z}_v.$$

The projection maps $x = (x_v)_v \mapsto x_v$ are continuous, i.e. the restricted product topology is finer than the product topology.

The group of idèles

The *idèles* are the subset of $\mathbb{A}$ defined by :

$$\mathbb{A}^\times = \prod_v (\mathbb{Q}_v^\times, \mathbb{Z}_v^\times) := \left\{ (x_v)_v \in \prod_v \mathbb{Q}_v^\times \; : \; \text{a.e. } x_v \in \mathbb{Z}_v^\times \right\}$$

It is a topological group with basis of open sets :

$$\prod_v U_v, \; U_v \text{ open of } \mathbb{Q}_v^\times \text{ and a.e. } U_v = \mathbb{Z}_v^\times .$$

Caution : This topology is finer than the induced topology of $\mathbb{A}$

Adelic notations

Spirit : adèles are manipulated place by place

For a finite set S of places of $\mathbb{Q}$, define :

- $\mathbb{A}_S = \prod_{v \in S} \mathbb{Q}_v \hookrightarrow \mathbb{A}$ the S-adèles
- $\mathbb{A}^S = \prod_{v \notin S} \mathbb{Q}_v \hookrightarrow \mathbb{A}$ the prime-to-S-adèles

Adelic norm $\|x\|_{\mathbb{A}} := \prod_v \|x_v\|_v$ (a.e. $\|x_v\|_v = 1$)

$\mathbb{A}$ is locally compact

Proof. For any S, let $X_S := \underbrace{\prod_{v \in S} \mathbb{Q}_v}_{\substack{\text{finite product} \\ \text{of loc. compact}}} \underbrace{\prod_{v \notin S} \mathbb{Z}_v}_{\substack{\text{product of} \\ \text{compact spaces}}}$

X_S are locally compact and cover X

$x \in X$ is $x \in X_S$ has a compact-open neighborhood $x \in K \subset X_S$

- K is also open in X (since open in X_S, which is open in X)
- K is also compact in X (embedded as $K \times \prod_{p \notin S} \mathbb{Z}_p$) □

Properties of the adèles

$\mathbb{A}$ is Hausdorff (i.e. the topology separates points)

Proof. Let $x \neq y \in \mathbb{A}$. Thus $x_v \neq y_v \in \mathbb{Q}_v$ for a v.

$\mathbb{Q}_v$ is Hausdorff thus there are disjoint open sets such that

$$X_v \ni x_v \text{ and } Y_v \ni y_v$$

$\pi_v^{-1}(X_v) = X_v \times \prod_{w \neq v} \mathbb{Z}_w$ and $\pi_v^{-1}(Y_v)$ are open and disjoint. □

Diagonal embedding of $\mathbb{Q}$

We have an embedding

$$\begin{array}{ccc} \mathbb{Q} & \hookrightarrow & \mathbb{A} \\ x & \mapsto & (x, x, x, \ldots) \end{array}$$

$\mathbb{Q}$ thus embedded are the *principal adèles*

Note : Indeed, $\mathbb{Q} \subset \mathbb{A}$ since $x \in \mathbb{Z}_v$ for almost all v

Product formula

Product formula

For all $x \in \mathbb{Q}^{\times}$, we have $\|x\|_{\mathbb{A}} = 1$.

Proof. By the fundamental theorem of arithmetics :

$$x = \frac{a}{b} = \prod_{i=1}^{r} p_i^{n_i}, \quad n_i \in \mathbb{Z}.$$

By definition,

$$\begin{cases} \|p^n\|_p & = & p^{-n} \\ \|p^n\|_\infty & = & p^n \end{cases}$$

Taking the product,

$$\|x\|_{\mathbb{A}} = \|x\|_\infty \prod_p \|x\|_p = \prod_{i=1}^{r} \|p_i^{n_i}\|_{p_i} \|p_i^{n_i}\|_\infty = 1. \quad \square$$

Properties of the adèles

$\mathbb{Q}$ is discrete in $\mathbb{A}$

Proof. $(\mathbb{A}, +)$ is an abelian group : enough to prove "0 isolated"

$$U := \{a \in \mathbb{A} \ : \ \|a\|_\infty < 1, \|a\|_p \leqslant 1\} = (-1,1) \times \prod_p \mathbb{Z}_p$$

By the product formula, $\|x\|_{\mathbb{A}} = \prod_v \|x\|_v \in \{0,1\}$ for $x \in \mathbb{Q}$

If moreover $x \in U$, then $\|x\|_\infty = 0$ thus $x = 0$.

We get $\mathbb{Q}^\times \cap U = \{0\}$. $\square$

Properties of the adèles

Approximation theorem

Let $a_1, \ldots, a_n \in \mathbb{Q}$ and $\varepsilon > 0$. Let $p_1, \ldots, p_n$ primes.

Then there is $x \in \mathbb{Q} \hookrightarrow \mathbb{A}$ with

$$\begin{aligned} &\|x - a_i\|_{p_i} \leqslant \varepsilon, \quad \forall i \in \{1, \ldots, n\} \\ &\|x\|_p \leqslant 1, \quad \forall p \neq p_i \end{aligned}$$

Proof. Split cases

- Suppose there is only $a_1 \in \mathbb{Z}$ which is nonzero. Write $a_1 = y + x$ with $y \in p_1^{e_1}\mathbb{Z}$ and $x \in p_2^{e_2} \cdots p_n^{e_n}\mathbb{Z}$, coprime ideals. Now $\|a_1 - x\|_1 = \|y\|_1 = p^{-e_1}$ and the others $\|x\|_i \leqslant 1$ are integers.
- Suppose $a_1, \ldots, a_n \in \mathbb{Z}$. We can approximate $(a_1, 0, \ldots)$ by x_1, $(0, a_2, \ldots)$ by x_2, etc. Then consider $x = x_1 + \cdots x_n \in \mathbb{Z}$.
- Suppose $a_1, \ldots, a_n \in \mathbb{Q}$. Multiply by a common denominator. Approach by $x_0 \in \mathbb{Z}$ in the previous case. Then divide back. □

Properties of the adèles

Approximation theorem

Let $a_1, \ldots, a_n \in \mathbb{Q}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$. Let $p_1, \ldots, p_n$ primes.

Then there is $x \in \mathbb{Q} \hookrightarrow \mathbb{A}$ with

$$\|x - a_i\|_{p_i} \leqslant \varepsilon_i, \quad \forall i \in \{1, \ldots, n\}$$
$$\|x\|_p \leqslant 1, \quad \forall p \neq p_i$$

Reformulate the conditions in the case of integers :

$$x \equiv a_i \mod p_i^{n_i}$$

This is the Chinese remainder theorem !

Approximation theorems are the analogous to the Chinese remainder theorem in the *adelic setting*

Opening towards approximation

$\mathbb{Q}$ discrete in $\mathbb{A}$

$\mathbb{Q}$ dense in $\mathbb{A}_S$ (weak approximation)

$\mathbb{Q}$ dense in $\mathbb{A}^S$ (strong approximation)

Further : All these are true for GL_n while S nontrivial

The quotient $\mathbb{A}/\mathbb{Q}$ is compact

Proof. Introduce the compact set

$$W := \{a \in \mathbb{A} \ : \ \|a\|_v \leqslant 1, \forall v\} = [-1,1] \times \prod_p \mathbb{Z}_p$$

Let $x \in \mathbb{A}$. We can approximate, x_v by $y_v \in \mathbb{Q}$ such that

$$\|x_p - y_p\|_p \leqslant 1$$
$$\|y_p\|_q \leqslant 1 \quad \forall q \neq p$$

Now, $\sum_p y_p \in \mathbb{Q}$ and translate by $y_\infty \in \mathbb{Z}$ to make it in $[-1,1]$. □

Another way : check the norm condition, for all v :

$$\|x-y\|_v = \left\| x_v - \sum_w y_w \right\|_v \leqslant \max\left(\|x_v - y_v\|_v, \max_{w \neq v} \|x_w\|_v \right) \leqslant 1. \quad \square$$

Properties of the adèles

The quotient $\mathbb{A}/\mathbb{Q}$ is compact

We have an action of $\mathbb{Q}$ on $\mathbb{A}$ by translations : $a \in \mathbb{A} \mapsto: a + x$

We just found a fundamental domain :

$$\mathbb{A}/\mathbb{Q} \simeq W := [0,1) \times \prod_p \mathbb{Z}_p$$

In other words,

$$\mathbb{A} = \bigsqcup_{k \in \mathbb{Q}} (k + W)$$

Conclusion : All in all $\mathbb{Q} \subset \mathbb{A}$ looks like $\mathbb{Z} \subset \mathbb{R}$

Properties of the idèles

Very similarly, we obtain a fundamental domain for the idèles :

$$\mathbb{A}^\times/\mathbb{Q}^\times \simeq D := (0, \infty) \times \prod_p \mathbb{Z}_p^\times$$

In other words,

$$\mathbb{A}^\times = \bigsqcup_{k \in \mathbb{Q}^\times} kD$$

Harmonic analysis

Locally compact groups

The good setting for harmonic analysis is a group which is :

- topological
- locally compact
- abelian
- Hausdorff

Jackpot : $\mathbb{A}$ is such a group, adequate for doing analysis

Harmonic analysis on a group G

Some classical definition in harmonic analysis :

A **character** is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$

Pontryagin dual $\widehat{G} = \mathrm{Hom}(G, S^1)$ group of *unitary* characters

- Operation : multiplication
- Topology : compact open i.e. $\{\chi \in \widehat{G} \ : \ \chi(K) \subseteq U\}$

The adèles are *self-dual*, and this allows nice Fourier analysis

Haar measure on G

We are particularly interested by invariant (*Haar*) measures on G :

$$\mathrm{d}(gx) = \mathrm{d}x \quad \text{i.e.} \quad \int_G f(gx)\mathrm{d}x = \int_G f(x)\mathrm{d}x$$

Existence

If G is locally compact then there exists a Haar measure.

It is unique up to a scalar.

If $\mathrm{d}x$ is a Haar measure on a ring R (invariant by $+$)

then $\mathrm{d}^\times x := \frac{\mathrm{d}x}{x}$ is a Haar measure on $R^\times$ (invariant by $\times$)

We normalize $\mathrm{d}x$ so that $\mathrm{vol}\,\mathbb{Z}_p = 1$

Reminder of Fourier theory on G

The *Fourier transform* of $f \in L^1(G)$ is defined by

$$\begin{array}{rcccl} \widehat{f} & : & \widehat{G} & \longrightarrow & \mathbb{C} \\ & & \chi & \longmapsto & \widehat{f}(\chi) = \int_G f(g)\overline{\chi}(g)\mathrm{d}g \end{array}$$

On $\mathbb{R}$, the characters are $\chi_y := x \mapsto e^{2i\pi xy} \in \widehat{\mathbb{R}}$ for $y \in \mathbb{R}$ and indeed we recover the classical Fourier transform :

$$\widehat{f}(\chi_y) =: \widehat{f}(x) = \int_G f(y)e^{-2i\pi xy}dy.$$

Fourier inversion formula : : there is a unique measure on $\widehat{G}$ (*Plancherel* measure $\mathrm{d}\chi$) such that for good f, $\widehat{\widehat{f}}(g) = f(-g)$, i.e.

$$f(g) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(g)\mathrm{d}\chi$$

Archimedean local integrals

Recall that $f_\infty(x_\infty) = e^{-\pi x_\infty^2}$ is self-dual : $\widehat{f_\infty} = f_\infty$

Introduce the archimedean *zeta integral*

$$\begin{aligned} Z(f_\infty, x_\infty) &:= \int_{\mathbb{R}^\times} f_\infty(x_\infty)|x_\infty|_\infty^s \mathrm{d}^\times x_\infty = 2\int_0^\infty e^{-\pi x_\infty^2}|x_\infty|_\infty^s \frac{\mathrm{d}x_\infty}{|x|_\infty} \\ &= \pi^{-\frac{s}{2}} \int_0^\infty e^{-u} u^{\frac{s}{2}} \frac{\mathrm{d}u}{u} = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}) =: \zeta\mathbb{R}(s) \end{aligned}$$

We obtain the archimedean factor as a zeta integral.

The p-adic analogue

Character on $\mathbb{Q}_p$: $e_p(p^n) := e^{2i\pi p^n}$: $e(\{p^n\}_p)$ for $n \in \mathbb{Z}$

What is the self-dual function in the p-adic setting ?

p-adic self-dual function

Let $f_p = 1_{\mathbb{Z}_p}$. Then $\widehat{f_p} = f_p$.

Proof. By definition, $\widehat{1}_{\mathbb{Z}_p}(y) = \int_{\mathbb{Z}_p} e_p(xy)\mathrm{d}x$

- If $y \in \mathbb{Z}_p$, then $e_p(xy) = 1$ and we get $\mathrm{vol}(\mathbb{Z}_p) = 1$

  If $y \notin \mathbb{Z}_p$, use the "translation trick" for $t \in \mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} e_p(xy)\mathrm{d}x = \int_{\mathbb{Z}_p} e_p((x+t)y)\mathrm{d}x = e_p(ty) \int_{\mathbb{Z}_p} e_p(xy)\mathrm{d}x$$

  so $\widehat{f_p}(y) = 0$. □

Non-archimedean local integrals

We have the decomposition

$$\mathbb{Z}_p = \bigsqcup_{x \in \mathbb{Z}/p^k\mathbb{Z}} x + p^k\mathbb{Z}_p$$

but $\mathrm{vol}(\mathbb{Z}_p) = 1$ and measure is translation-invariant so that

$$\mathrm{vol}(p^k\mathbb{Z}_p) = p^{-k}$$

We have the decomposition

$$\mathbb{Z}_p^\times = \bigsqcup_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} x + p\mathbb{Z}_p$$

but $\mathrm{vol}(\mathbb{Z}_p) = 1$ and measure is translation-invariant so that

$$\mathrm{vol}(\mathbb{Z}_p^\times) = \frac{p-1}{p}$$

We normalize $\mathrm{d}^\times x = \frac{p}{p-1} \frac{\mathrm{d}x}{|x|_p}$ so that $\mathrm{vol}_\times(\mathbb{Z}_p^\times) = 1$

Non-archimedean local integrals

Tip to compute p-adic integrals : slice by valuations (functions are locally constant, so integrals are sums)

$$\mathbb{Z}_p \backslash \{0\} = \bigsqcup_{n \geqslant 0} p^n \mathbb{Z}_p^\times$$

Non-archimedean local integrals

Recall that $f_p = 1_{\mathbb{Z}_p}$ is self-dual : $\widehat{f_p} = f_p$

Introduce the p-adic zeta integral

$$\begin{aligned} Z(f_p, s) &:= \int_{\mathbb{Q}_p^\times} f_p(x_p)|x_p|_p^s \mathrm{d}^\times x_p = \int_{\mathbb{Z}_p\setminus\{0\}} |x_p|_p^s \mathrm{d}^\times x_p \\ &= \sum_{k\geqslant 0} \int_{p^k\mathbb{Z}_p^\times} |x_p|_p^s \mathrm{d}^\times x_p = \frac{p}{p-1} \sum_{k\geqslant 0} \int_{p^k\mathbb{Z}_p^\times} |x_p|_p^{s-1} \mathrm{d}x_p \\ &= \frac{p}{p-1} \sum_{k\geqslant 0} p^{-k(s-1)} \mathrm{vol}(p^k\mathbb{Z}_p^\times) \\ &= \frac{p}{p-1} \sum_{k\geqslant 0} p^{-k(s-1)} \frac{p-1}{p} p^{-k} \\ &= \sum_{k\geqslant 0} p^{-ks} = (1-p^{-s})^{-1} := \zeta_p(s) \end{aligned}$$

We obtain the p-factor as zeta integral

Tate's thesis

Adelic Mellin transform

Let $f_{\mathbb{A}} = \prod_v f_v$ the adelic self-dual function

Introduce the adelic zeta integral

$$Z(f_{\mathbb{A}}, s) = \int_{\mathbb{A}^\times} f_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s \mathrm{d}^\times x = \prod_v \int_{\mathbb{Q}_v^\times} f_v(x_v)|x_v|_v^s \mathrm{d}^\times x_v$$

$$= \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}) \prod_p (1 - p^{-s})^{-1} = \zeta_{\mathbb{R}}(s)\zeta(s) = \Lambda(s)$$

The adelic zeta integral directly appears as the completed ζ

Another proof of the Euler product

Recall $\mathbb{A}^\times/\mathbb{Q}_+^\times \simeq \mathbb{R}^\times \times \prod_p \mathbb{Z}_p^\times =: \mathbb{R}^\times \times \widehat{\mathbb{Z}}^\times$

We can cut the adelic integral by classes :

$$\begin{aligned} Z(f_\mathbb{A}, s) &= \sum_{k \in \mathbb{Q}_+^\times} \int_{\mathbb{R}^\times \times k\hat{\mathbb{Z}}^\times} f(x)|x|_\mathbb{A}^s \mathrm{d}^\times x = \sum_{k \geqslant 1} \int_{(\mathbb{R}^\times \times k\hat{\mathbb{Z}}^\times)} f(x)|x|_\mathbb{A}^s \mathrm{d}^\times x \\ &= \sum_{k \geqslant 1} \|k\|_{\mathbb{A}_f}^s \int_{\mathbb{R}^\times} f_\infty(x_\infty)|x_\infty|_\infty^s \mathrm{d}^\times x_\infty = \sum_{k \geqslant 1} k^{-s} \int_{-\infty}^{\infty} e^{-\pi t^2} |t|^s \frac{\mathrm{d}t}{t} \\ &= \zeta(s)\pi^{-s/2}\Gamma(\tfrac{s}{2}) \end{aligned}$$

This only uses the Dirichlet series definition of ζ

It *proves* the Euler product decomposition

Note : the fundamental theorem of arithmetic lurks behind !

Adelic Poisson summation formula

Adelic Poisson summation formula

For $f \in \mathcal{S}(\mathbb{A})$, we have $\displaystyle\sum_{k\in\mathbb{Q}} f(k) = \sum_{k\in\mathbb{Q}} \widehat{f}(k)$

Proof. The proof mimics the classical case. Let $F(x) := \sum_{\ell\in\mathbb{Q}} f(x+\ell)$ and consider a $\mathbb{Q}$-invariant character ψ of $\mathbb{A}$

$$\widehat{F}(k) = \int_{\mathbb{A}/\mathbb{Q}\simeq W} \left(\sum_{\ell\in\mathbb{Q}} f(x+\ell)\right) \psi(kx)\mathrm{d}x = \sum_{\ell\in\mathbb{Q}} \int_W f(x+\ell)\psi(kx)\mathrm{d}x$$

$$= \sum_{\ell\in\mathbb{Q}} \int_{W+\ell} f(x)\psi(k(x-\ell))\mathrm{d}x = \int_{\mathbb{A}} f(x)\psi(kx)\mathrm{d}x = \widehat{f}(k)$$

Now, $F(x) = \displaystyle\sum_{k\in\mathbb{Q}} \widehat{f}(k)\overline{\psi}(kx)$ and... evaluate at $x = 0$. □

Adelic theta function

Introduce $\Theta(x) = \sum_{k\in\mathbb{Q}} f(kx)$ the adelic theta

By applying Poisson, $\Theta(x) = \frac{1}{|x|}\Theta(\frac{1}{x})$ for all $x \in \mathbb{A}^\times$

By cutting the integral over $\mathbb{A}^\times$ by $\mathbb{Q}^\times$-classes,

$$Z(f_{\mathbb{A}}, s) = \int_{\mathbb{A}^\times} f(x)|x|_{\mathbb{A}}^s \mathrm{d}^\times x = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \sum_{k\in\mathbb{Q}^\times} f(kx)|kx|_{\mathbb{A}}^s \mathrm{d}^\times x$$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \widetilde{\Theta}(x)|x|_{\mathbb{A}}^s \mathrm{d}^\times x$$

After regularizing properly (adding the zero term),

$$Z(f_{\mathbb{A}}, s) + f(0)\int |x|_{\mathbb{A}}^s d^\times x = Z(f_{\mathbb{A}}, 1-s) + f(0)\int |x|_{\mathbb{A}}^{1-s} d^\times x$$

We obtain the analytic continuation and functional equation.

This is **Tate's thesis**

Towards automorphic forms

The spirit of automorphic forms

Automorphic forms (loosely)

For a "nice" algebraic group G, and automorphic form is

$$\phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that

- **automorphy :** $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(\mathbb{Q})$
- **central action :** $\phi(zg) = \omega(z)\phi(g)$ for all $z \in Z(\mathbb{A})$
- **moderate growth**

Here, ω is a character on $Z(\mathbb{A})$ and Z is the center of G.

Question : What is the relation with classical objects ?

The case of $\mathrm{GL}(1)$

Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

Hecke character $\omega : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ continuous

Adelic-classical correspondence

$$\left\{ \begin{array}{c} \omega : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times \\ \text{adelic Hecke} \\ \text{characters} \\ \text{of finite order} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \\ \text{primitive classical} \\ \text{Dirichlet characters} \end{array} \right\}$$

These are the automorphic forms on $\mathrm{GL}(1, \mathbb{A}) = \mathbb{A}^\times$

The case of $\mathrm{GL}(2)$

Modular form $\phi : \Gamma\backslash\mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}^\times$

Automorphic form $f : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K \to \mathbb{C}^\times$

Adelic-classical correspondence

$$\left\{ \begin{array}{c} f : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K \to \mathbb{C}^\times \\ \text{modular forms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \phi : \Gamma\backslash\mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}^\times \\ \text{automorphic forms} \end{array} \right\}$$

These are the automorphic forms on $\mathrm{GL}(2, \mathbb{A})$

The adèles $\mathbb{A}$ are the perfect setting to study them

Adèles are fancy... but also have many advantages

- adèles are easy (analytic properties, place by place, etc.)
- replacing $\mathbb{Z}$ by $\mathbb{Q}$ (e.g. conjugacy classes in $\mathrm{GL}_2(\mathbb{Q})$ easier)
- analysis on $\mathbb{Q}$-points (e.g. $\mathrm{GL}_n(\mathbb{Q})$ discrete lattice in $\mathrm{GL}_n(\mathbb{A})$)
- there is no Poisson formula on $\mathbb{Q}_p$, but there is on $\mathbb{A}$
- the prime structure is embedded in the adelic structure : objects x (essentially) split into $\prod_v x_v$ (e.g. Maass forms or L-functions)
- Unification of finite and archimedean places : Hecke opeators T_p and Laplacian Δ play a symmetric role
- proofs get technical with higher degree (characters, groups of units, class group), but remain identical in the adelic setting (e.g. Hilbert modular forms, trace formulas...) : the packaging looks perfect
- Hecke and Tate did it for $\mathrm{GL}(1)$, Godement and Jacquet did it for $\mathrm{GL}(n)$, opening a wide door to the Langlands program
- It allows to study automorphic representations

Extra fruits from Tate's thesis

- works for any number field K
- finiteness of the class group
- Dirichlet theorem of units
- we can consider more general zeta integrals

$$Z(f, \chi, s) := \int_{\mathbb{A}^\times} f(x)\chi(x)|x|_{\mathbb{A}}^s \mathrm{d}^\times x$$

  They also satisfy analytic continuation, functional equation

- $L_p(\chi, s)$ is "the" common divisor of the $Z_p(f, \chi, s)$
- local L-functions have functional equations
- global ε-factors factorize as product of local ε-factors

Thank you !

非常感谢 !